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Stokes' theorem, gauge symmetry and the time-dependent Aharonov-Bohm effect

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Stokes' theorem is investigated in the context of the time-dependent Aharonov-Bohm effect—the two-slit quantum interference experiment with a *time varying* solenoid between the slits. The time varying solenoid produces an electric field which leads to an additional phase shift which is found to exactly cancel the time-dependent part of the usual magnetic Aharonov-Bohm phase shift. This electric field arises from a combination of a non-single valued scalar potential and/or a 3-vector potential. The gauge transformation which leads to the scalar and 3-vector potentials for the electric field is non-single valued. This feature is connected with the non-simply connected topology of the Aharonov-Bohm set-up. The non-single valued nature of the gauge transformation function has interesting consequences for the 4-dimensional Stokes' theorem for the time-dependent Aharonov-Bohm effect. An experimental test of these conclusions is proposed. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4869324>]

I. INTRODUCTION

In this work, we investigate the interplay between Stokes' theorem and gauge symmetry in the context of the *time varying* Aharonov-Bohm (AB) effect.^{1,2} Already the static AB effect—placing an infinite magnetic flux carrying solenoid between the slits of a quantum mechanical two-slit experiment—shows a deep interrelation between gauge symmetry and the 3-dimensional Stokes' theorem. If one allows the current and therefore the magnetic flux through the solenoid to be time dependent then one needs to take into account both the electric field (generated from $\mathbf{E} = -\partial_t \mathbf{A}$) as well as magnetic field (generated from $\mathbf{B} = \nabla \times \mathbf{A}$). The introduction of time dependence means that one must consider space-time coordinates and differentials (i.e., $x^\mu = (t, \mathbf{x})$ and $dx^\mu = (dt, d\mathbf{x})$) in doing the relevant integrals rather than simply spatial coordinates and differentials (i.e., $x^i = \mathbf{x}$ and $dx^i = d\mathbf{x}$). One needs to consider Stokes' theorem in 4-dimensional Minkowski space-time. We will show that the electric field coming from the time-dependent 3-vector potential, associated with the time-dependent magnetic flux in the solenoid, can be written entirely in terms of the 3-vector potential, $\mathbf{A}$, or in terms of a scalar potential, ϕ , or some combination of the two. The relationship between these different ways of writing the electric field is through a gauge transformation $A^\mu \rightarrow A^\mu + \partial^\mu \chi$. However, the gauge transformation function, χ , and the scalar potential, ϕ , are found to be non-single valued. This has interesting consequences for Stokes' theorem for this case. This analysis also shows that the phase shift coming from the electric field exactly cancels the time varying part of magnetic AB phase *during the period when the potential is being varied with respect to time*. Thus, one can experimentally test the conclusions in this work. Although there has been ample experimental confirmation of the static Aharonov-Bohm effect^{3,4} there has been no definitive experimental test of the time-dependent Aharonov-Bohm effect to date. The one test that has been performed^{5,6} supports the conclusions presented here that the AB phase shift does not inherit the time dependence of the magnetic flux.

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II. STOKES' THEOREM USING DIFFERENTIAL FORMS

We begin by reviewing Stokes' theorem in 3- and 4-dimensions. In 3-vector notation the 3-dimensional Stokes' theorem is given by

$$\oint_{\partial S} \mathbf{A} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}, \quad (1)$$

where $\mathbf{A}$ is a 3-vector field and the first integral is a closed line integral and the second is an area integral over S whose boundary is ∂S . In differential forms notation, Stokes' theorem takes the following elegant form

$$\oint_{\partial c} \omega = \int_c d\omega, \quad (2)$$

(for a brief review of differential forms at the level needed here see Ref. 7). In (2) ω is a p -form, $d\omega$ (the exterior derivative of ω) is a $p + 1$ -form, c is a $p + 1$ chain, and ∂c is the boundary of c , i.e., a p chain. In this work, we will apply Stokes' theorem to electromagnetism so for our p -form ω we take the 1-form vector potential

$$\omega = A = A_\mu dx^\mu = \phi dt - \mathbf{A} \cdot d\mathbf{x}. \quad (3)$$

We have given the 1-form A in 4-vector and 3-vector notation. Throughout the paper we set $c = 1$. The exterior derivative of the 1-form A is the Faraday 2-form F^7 where

$$\begin{aligned} F = dA &= -\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy. \end{aligned} \quad (4)$$

We have written the Faraday 2-form in 4-vector and 3-vector notation. In (4) we have used the wedge products like $dx \wedge dt$, $dy \wedge dz$ which are anti-symmetric under exchange of the differentials.

With this differential form notation one can write down the expression for the usual static, magnetic AB phase shift. In the case where one has a solenoid with a static current and magnetic field, the 4-vector potential becomes $A_\mu = (0, \mathbf{A})$, i.e., the scalar potential is zero. Thus the phase AB shift, $\delta\alpha_{AB}$, for this case^{1,2} becomes

$$\delta\alpha_{AB} = \frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{x} = \frac{e}{\hbar} \int \mathbf{B} \cdot d\mathbf{S}, \quad (5)$$

where e is the charge of the particle. In arriving at (5) we have used Stokes' theorem (2) and the fact that $B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy = \mathbf{B} \cdot d\mathbf{S}$. Note that the AB phase shift, $\delta\alpha_{AB}$, can be written equivalently as either the closed path integral of $\mathbf{A}$ or the surface integral of the magnetic field, i.e., the curl of $\mathbf{A}$.

III. TIME-DEPENDENT AHARONOV-BOHM EFFECT

Now we use the results of Sec. II to address the case when the current and magnetic flux through the solenoid vary with time. To this end we need to write down the 4-vector potential for the solenoid. The 3-vector potential inside and outside the solenoid (which is taken to have a radius $\rho = R$) is

$$\begin{aligned} \mathbf{A}_{\text{in}} &= \frac{\rho B(t)}{2} \hat{\phi} \text{ for } \rho < R, \\ \mathbf{A}_{\text{out}} &= \frac{B(t)R^2}{2\rho} \hat{\phi} \text{ for } \rho \geq R, \end{aligned} \quad (6)$$

and the scalar potential is normally taken as zero everywhere, i.e., $\phi = 0$. We will come back to this gauge choice of $\phi = 0$ shortly. The magnetic field, $B(t)$, now depends on time since the current of the solenoid is being varied. We could write this time dependent magnetic field in terms of the time dependent current through the solenoid, $I(t)$, and the number of turns per unit length of the solenoid, N , as $B(t) \propto NI(t)$. However, for our purposes the time dependence can just be left in terms of the

time variation of the amplitude of the magnetic field. Taking the curl of the 3-vector potential in (6) yields the magnetic field

$$\begin{aligned}\mathbf{B}_{\text{in}} &= \nabla \times \mathbf{A}_{\text{in}} = B(t)\hat{\mathbf{z}} \text{ for } \rho < R, \\ \mathbf{B}_{\text{out}} &= \nabla \times \mathbf{A}_{\text{out}} = 0 \text{ for } \rho \geq R,\end{aligned}\quad (7)$$

the only difference from the static case is now the magnetic field inside the solenoid is time varying. The magnetic field outside is still zero. The new feature resulting from allowing the magnetic flux to vary with time is that there is an electric field coming from $\mathbf{E} = -\partial_t \mathbf{A}$. Explicitly using (6) we find

$$\begin{aligned}\mathbf{E}_{\text{in}} &= -\frac{\partial \mathbf{A}_{\text{in}}}{\partial t} = -\frac{\rho \dot{B}(t)}{2} \hat{\phi} \text{ for } \rho < R, \\ \mathbf{E}_{\text{out}} &= -\frac{\partial \mathbf{A}_{\text{out}}}{\partial t} = -\frac{\dot{B}(t)R^2}{2\rho} \hat{\phi} \text{ for } \rho \geq R,\end{aligned}\quad (8)$$

where the overdots are derivatives with respect to time. One point to note is that while (6)–(8) assume arbitrary time dependence for the flux, $B(t)$, there is actually some restriction coming from Maxwell's equations that must be taken into account. While Faraday's Law $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ is consistent with an arbitrary time dependence for $B(t)$, the sourceless Ampere-Maxwell equation, $\nabla \times \mathbf{B} = \partial_t \mathbf{E}$ only works with the expressions in Eqs. (7) and (8) if the flux is linearly dependent on time, i.e.

$$B(t) = B_0 + B_1 t, \quad (9)$$

where B_0, B_1 are constants. Later in Sec. IV, we will show that it is possible to consider other variations of flux other than the linear dependence of (9). This is accomplished by changing the spatial dependence from ρ ($\frac{1}{\rho}$) for the fields inside (outside) the solenoid. For the sinusoidal dependence considered in Sec. IV we find that the spatial dependence will be ordinary Bessel functions. However, for this section it is good to keep in mind that strictly the time dependence of the flux is that given in Eq. (9). It turns out that this linear time dependence, although unphysical for arbitrary times (there is some practical limit to how large of a B-field one can make) it is good for illuminating the unique features of the time dependent Aharonov-Bohm effect.

We now evaluate the AB phase shift using the surface area integral of the fields, i.e., $\int_c F$ where F is the Faraday 2-form (4). The difference from the static case, aside from the time variation of the magnetic field in (7), is that there is a time dependent electric field (8) which contributes to the phase. The AB phase in terms of the Faraday 2-form (i.e., in terms of the electric and magnetic fields) is

$$\begin{aligned}\delta\alpha_{\text{AB}} &= \frac{e}{\hbar} \int_c F = \frac{e}{\hbar} \int (E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \\ &= -\frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{x} + \frac{e}{\hbar} \int \mathbf{B} \cdot d\mathbf{S} = -\frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{x} + \frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{x} = 0.\end{aligned}\quad (10)$$

In the second line we have written the three electric field terms from the first line as $-\oint \mathbf{A} \cdot d\mathbf{x}$ using $\mathbf{E} = -\partial_t \mathbf{A}$ and performing the dt integration. Next the three magnetic terms of the first line were converted from differential form notation to 3-vector notation, $\int \mathbf{B} \cdot d\mathbf{S}$. Then using $\mathbf{B} = \nabla \times \mathbf{A}$ and the 3-vector form of Stokes' theorem we end up with $+\oint \mathbf{A} \cdot d\mathbf{x}$. Thus, we find that the electric and magnetic contributions in (10) exactly cancel giving the prediction of *no time-dependent phase shift* for the time-dependent Aharonov-Bohm effect. We should clarify that this cancellation is only in effect for the time-dependent part of the magnetic field. If the vector potential can be split into time-independent and time-dependent parts $\mathbf{A} = \mathbf{A}_0(\mathbf{x}) + \mathbf{A}_1(\mathbf{x}, t)$ then the magnetic field also splits into time-independent and time-dependent parts $\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{B}_0(\mathbf{x}) + \mathbf{B}_1(\mathbf{x}, t)$. The electric field only comes from $\mathbf{A}_1(\mathbf{x}, t)$ and thus only the time-dependent magnetic field $\mathbf{B}_1(\mathbf{x}, t)$ part is canceled by the electric field. Initially, one might expect that the AB phase in (10) would be time-dependent with the time-dependence coming from the time-varying magnetic flux. Instead, we find that, at least according to the Faraday 2-form/E& M field expression for the AB phase, that the AB phase

is time-independent due to the cancellation of the electric and magnetic field parts. This is exactly what was found in Ref. 8 where it was shown that the standard time-dependent AB phase shift due to the magnetic field was canceled by the phase shift coming from the electric force on the electrons. The electric field would accelerate/decelerate the electrons thus changing their position with respect to the case when there was no electric field outside the solenoid, i.e., for a static magnetic flux. This shift in position, associated with the electric field, would give a corresponding shift in phase.

Since we are considering time dependence of the flux one encounters the questions of “what surface?” and “at what time?” are meant in Eq. (10). To address these questions we look at the change in phase along some infinitesimal path length $\Delta \mathbf{x}$ and some associated infinitesimal area $\Delta \mathbf{S}$, and show that for these infinitesimal temporal and geometrical quantities that one gets cancellation between the electric contribution in (10) and the *time dependent* contribution of the magnetic field. Thus, adding up/integrating these infinitesimal quantities leads to the result in Eq. (10). For an infinitesimal time interval, Δt , length interval, $\Delta \mathbf{x}$, and area interval, $\Delta \mathbf{S}$, one can write (10) as

$$\begin{aligned}\Delta(\delta\alpha_{AB}) &= \frac{e}{\hbar} (\mathbf{E} \cdot \Delta \mathbf{x} \Delta t + \Delta \mathbf{B} \cdot \Delta \mathbf{S}) \\ &= \frac{e}{\hbar} \left(-\frac{R^2}{2\rho} (\dot{B} \Delta t) (\rho \Delta \varphi) + \frac{\Delta \varphi R^2}{2} (\dot{B} \Delta t) \right) = 0.\end{aligned}\quad (11)$$

In (11) we have Taylor expanded $B(t) = B_0 + \dot{B} \Delta t + \mathcal{O}(\Delta t)^2$. The B_0 term gives the usual, static part of the Aharonov-Bohm phase shift; the focus of this paper is the time-dependent contribution which to first order in Δt is given by the second term in the Taylor expansion for $B(t)$. In (11) we have also used $\Delta \mathbf{x} = (\rho \Delta \varphi) \hat{\phi}$, $\Delta \mathbf{S} = \frac{1}{2} (R \Delta \varphi) R \hat{z} = \frac{1}{2} \Delta \varphi R^2 \hat{z}$, and the expression for $\mathbf{E}_{\text{out}}$ from (8). Adding up/integrating all these infinitesimal phase shifts, with $\Delta(\delta\alpha_{AB}) = 0$, gives the result $\delta\alpha_{AB} = 0$ in Eq. (10), i.e., the time-dependent part of the phase shift cancels. The cancellation between the electric and magnetic contributions to the phase in (11) is made more transparent in the case when the flux varies linearly as in (9). In this case one can replace Δt by a finite time interval T and the electric contribution in (11) takes the form

$$-\frac{e}{\hbar} \left(\frac{R^2}{2\rho} (B_1 T) (\rho \Delta \varphi) \right) = -\frac{e}{\hbar} \left(\frac{R^2 \Delta \varphi}{2} (B_1 T) \right).$$

The magnetic contribution takes the form

$$+\frac{e}{\hbar} \left(\frac{R^2 \Delta \varphi}{2} (B_0 + B_1 T) \right).$$

Comparing these two expressions one finds that the time dependent pieces (i.e., the terms $\propto B_1 T$) cancel leaving only the static contribution (i.e., the term $\propto B_0$).

However the above analysis, as well as that in Ref. 8, leaves open the question as to the status of Stokes’ theorem in the time-dependent case in regard to the one-form side of Eq. (2). At first glance it would seem Stokes’ theorem is violated in this case which would then call into question the above analysis. In the above we have calculated the AB phase $\delta\alpha_{AB}$, through the right-hand side of Stokes’ theorem as given in (2) using the Faraday 2-form $d\omega = dA = F$ from (4). The result was $\delta\alpha_{AB} = 0$. We now calculate the left-hand side of Stokes’ theorem as given in (2) using the 4-vector potential 1-form $\omega = A$ (3) and at first find that, apparently, $\delta\alpha_{AB} \neq 0$ which would imply a violation of the 4-dimensional Stokes’ theorem. In the end, we resolve this through the non-simply connected topology of the Aharonov-Bohm set-up and the associated non-single valued gauge potentials.

The 4-vector potential in this case is given by $A^\mu = (\phi, \mathbf{A}) = (0, \mathbf{A}(t, \mathbf{x}))$ where the 3-vector potential is given by (6) and the scalar potential is zero. This *gauge* choice for A^μ does give the correct magnetic (7) and electric fields (8) for this situation. But we will see that there are other gauges for which the value of $\int_c \omega = \int A$ will depend on the gauge due to the non-single valued nature of the gauge transformation function which is connected with the fact that the space in this case is *non-simply connected* (see pp. 102–103 of Ref. 7 for a discussion on this point). Using

$\phi = 0$ and $\mathbf{A}(t, \mathbf{x})$ from (6) we apparently find that the AB phase shift in terms of A^μ is

$$\begin{aligned}\delta\alpha_{AB} &= -\frac{e}{\hbar} \int_{\partial c} \omega = -\frac{e}{\hbar} \oint A_\mu dx^\mu = -\frac{e}{\hbar} \left(\int \phi dt - \oint \mathbf{A}(t, \mathbf{x}) \cdot d\mathbf{x} \right) = \frac{e}{\hbar} \oint \mathbf{A}(t, \mathbf{x}) \cdot d\mathbf{x} \\ &= \frac{e}{\hbar} \int \nabla \times \mathbf{A}(t, \mathbf{x}) \cdot d\mathbf{S} = \frac{e}{\hbar} \int \mathbf{B}(t, \mathbf{x}) \cdot d\mathbf{S},\end{aligned}\quad (12)$$

where in going from the end of the first line to the second line we have used the Stokes' theorem in the form (1) and then used $\nabla \times \mathbf{A} = \mathbf{B}$. The time dependence of the AB phase shift in (12) is exactly same as the magnetic part of the AB phase shift given in (10) where the phase shift was calculated using the Faraday 2-form. Thus, from the 4-vector potential calculation in (12) it appears that we should get the usual magnetic AB phase shift but with a time dependence coming from the time dependence of the magnetic field. This is what was predicted in earlier work on the time-dependent AB effect.⁹ But in this way one does not take into account the effect of the electric field. In the work,¹⁰ time dependent Berry Phases were briefly considered and it was noted that in the case of time-dependent geometric fluxes that there would be a "motive force" similar to the electromotive force in Faraday's law. However in this paper we emphasize that our analysis shows an exact cancellation of the "magnetic" and "electric" contributions to the time-dependent part of the AB phase shift.

To get an "electric" and "magnetic" cancellation of the potentials, ϕ , $\mathbf{A}$ one would need a non-zero scalar potential. In fact, one *can* find a gauge transformation which does give scalar and 3-vector potentials which give the magnetic and electric fields from (7) and (8) but for which $\phi \neq 0$. We begin by noting that one can get the outside magnetic and electric fields from (7) and (8) by taking the scalar potential $\phi_{\text{out}} = R^2 \dot{B}(t) \varphi / 2$ and $\mathbf{A}_{\text{out}} = 0$. Taking $\mathbf{E}_{\text{out}} = -\nabla \phi_{\text{out}} - \partial_t \mathbf{A}_{\text{out}}$ does give $\mathbf{E}_{\text{out}} = -\frac{\dot{B}(t)R^2}{2\rho} \hat{\varphi}$ and taking $\mathbf{B}_{\text{out}} = \nabla \times \mathbf{A}_{\text{out}}$ does give $\mathbf{B}_{\text{out}} = 0$. It will be noticed that the scalar potential is non-single valued due to the presence of the angular coordinate φ -dependence. Such non-single valued functions cannot exist (or are pathological) in simply connected spaces, but the Aharonov-Bohm setup is *non-simply* connected. Because of this non-single value functions can be considered (see the discussion on pp. 102–103 of Ref. 7). The non-single valued form of the potentials is connected to the form of the gauge potentials given in (6) by the following gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi; \quad \chi = \frac{KR^2 B(t)\varphi}{2}, \quad (13)$$

where K is some constant in the range $0 \leq K \leq 1$. Note that the gauge function χ is also non-single valued and this is again connected with the fact that the space in the Aharonov-Bohm setup has non-simply connected topology. With this gauge transformation (13) the outside gauge potentials from (6) become

$$\phi_{\text{out}} = K \frac{R^2 \dot{B}(t)\varphi}{2}; \quad \mathbf{A}_{\text{out}} = (1 - K) \frac{R^2 B(t)}{2\rho} \hat{\varphi}. \quad (14)$$

When $K = 0$ the outside electric field is given purely in terms of the non-zero, single-valued 3-vector potential, and when $K = 1$ the outside electric field is given purely in terms of the non-zero, non-single-valued scalar potential. For K at intermediate values the outside electric field $\mathbf{E} = -\nabla \phi_{\text{out}} - \partial_t \mathbf{A}_{\text{out}}$ comes from a combination of the scalar and 3-vector potential. In all cases the outside magnetic field is zero. One important thing to point out is that the quantity $\oint A_\mu dx^\mu$ is no longer gauge invariant due to the non-single valued character of the gauge function χ from (13) which is related to the non-simply connected topology of the Aharonov-Bohm set-up. Explicitly, under the gauge transformation (13) we find

$$\oint A_\mu dx^\mu \rightarrow \oint A_\mu dx^\mu + \oint \partial_\mu \chi dx^\mu = \oint A_\mu dx^\mu + (\chi(f) - \chi(i)), \quad (15)$$

where $\chi(f) - \chi(i)$ is the difference between the final and initial point of the path. For a closed path and a single value χ this will be zero and $\oint A_\mu dx^\mu$ will be gauge invariant. However, for a non-single value gauge function $\chi(f) - \chi(i)$ will not be zero for a closed path and $\oint A_\mu dx^\mu$ is *not*

gauge invariant. Note that the AB phase shift as given in terms of the Faraday 2-form (10) is still gauge invariant since the electric and magnetic fields as well as the “area” $dx^\mu \wedge dx^\nu$ are the same regardless of whether the fields come from a single valued 3-vector potential as in (6), a non-single valued scalar potential $\phi_{\text{out}} = R^2 B(t)\varphi/2$ or some combination of the two as in (14). The question arises is there a gauge (i.e., a choice of K in (13)) for which $\oint A_\mu dx^\mu = -\frac{1}{2} \int F_{\mu\nu} dx^\mu \wedge dx^\nu$? The answer is “yes” for $K = \frac{1}{2}$. For this choice we have

$$\phi_{\text{out}} = \frac{R^2 \dot{B}(t)\varphi}{4}; \quad \mathbf{A}_{\text{out}} = \frac{R^2 B(t)}{4\rho} \hat{\phi}. \quad (16)$$

For this gauge choice of the potentials the electric field is seen to come equally from ϕ_{out} and $\mathbf{A}_{\text{out}}$ (16)

$$\mathbf{E} = -\nabla\phi_{\text{out}} - \partial_t \mathbf{A}_{\text{out}} = -\frac{R^2 \dot{B}(t)}{4\rho} - \frac{R^2 \dot{B}(t)}{4\rho} = -\frac{R^2 \dot{B}(t)}{2\rho}.$$

For the gauge choice, $K = 1/2$, one can see that the contributions of ϕ_{out} and $\mathbf{A}_{\text{out}}$ to $\oint A_\mu dx^\mu$ cancel thus bringing the gauge potential expression for the AB phase shift into agreement with the Faraday 2-form expression for the AB gauge. For the interval Δt the infinitesimal contributions from the scalar and vector potentials in (16) become

$$\Delta\phi\Delta t - \Delta\mathbf{A} \cdot \Delta\mathbf{x} = \frac{R^2 \Delta B \Delta\varphi}{4\Delta t} \Delta t - \frac{R^2 \Delta B}{4\rho} \rho \Delta\varphi = 0,$$

where we have used $d\mathbf{x} \rightarrow (\rho\Delta\varphi)\hat{\phi}$. Adding up (i.e., integrating) these infinitesimal contributions from (17) gives $\oint A_\mu dx^\mu = 0$ and we find that the 4-dimensional Stokes’ theorem (i.e., $\oint A_\mu dx^\mu = -\frac{1}{2} \int F_{\mu\nu} dx^\mu \wedge dx^\nu$) is now satisfied but only for the gauge $K = \frac{1}{2}$. In the arguments above, this gauge dependence of the 4-dimensional Stokes’ theorem can be traced to the gauge dependence of $\oint A_\mu dx^\mu$ which arises from the non-single valued character of the gauge transformation function χ in (13). This in turn is connected with the non-simply connected topology of the Aharonov-Bohm set-up. We note that the Faraday 2-form side of the 4-dimensional Stokes’ theorem, $-\frac{1}{2} \int F_{\mu\nu} dx^\mu \wedge dx^\nu$, is gauge invariant even for the non-single valued gauge function χ from (13). Thus, one can experimentally test the correctness (or not) of the above arguments. If one performs the time-dependent AB experiment and finds that the phase shift, $\delta\alpha_{\text{AB}}$, is not time-dependent then the above arguments are correct; if one performs the time-dependent AB experiment and finds that the phase shift, $\delta\alpha_{\text{AB}}$, inherits the time dependence of the magnetic field/magnetic flux then the above arguments are not correct. However, if this last case is the one selected by experiment one needs to understand why the electric field in the time-dependent case has no influence on the phase shift as given by (10).

IV. SINUSOIDAL FLUX VARIATION

Although up to now we have assumed an arbitrary time variation for the flux, $B(t)$, as already mentioned, due to the restriction coming from Maxwell-Ampere’s Law (i.e., $\nabla \times \mathbf{B} = \partial_t \mathbf{E}$) one is implicitly dealing with only a linearly increasing flux as given in Eq. (9). While this linearly increasing flux is good for illustrating the basic features of the time dependent Aharonov-Bohm effect (in particular, the claimed cancellation between the electric and magnetic contributions to the phase shift given in (10) or (11)) one might ask if more general time variations can be considered, and if so does one still have the same cancellations of the electric and magnetic contributions to the phase shift. In this section, we show this is possible for the physically realistic case of sinusoidally varying fields. The reason to focus on sinusoidally varying fields is that they would be the ones most likely used experimentally to test the predictions made in this paper. To begin we will assume a more general ρ dependence for the vector potential. Previously in (6) we had taken the ρ dependence of $\mathbf{A}$ as $\propto \rho$ and $\propto 1/\rho$ for inside and outside the solenoid, respectively. Here, we assume a vector potential of the form

$$\mathbf{A} = F(\rho)e^{i\omega t} \hat{\phi}, \quad (17)$$

where $F(\rho)$ is some function of ρ and we have already put in the assumed sinusoidal time dependence with frequency ω . The direction of $\mathbf{A}$ is still taken to be in the $\hat{\phi}$ direction. Using this form of the vector potential in (17) to calculate the magnetic and electric fields via $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\partial_t \mathbf{A}$ and then inserting these in $\nabla \times \mathbf{B} = \partial_t \mathbf{E}$ we arrive at the following equation for $F(\rho)$

$$F''(\rho) + \frac{F'(\rho)}{\rho} - \frac{F(\rho)}{\rho^2} + \omega^2 F(\rho) = 0, \quad (18)$$

where primes denote differentiation with respect to ρ . Changing variables to $x = \omega\rho$ Eq. (18) becomes

$$F''(x) + \frac{F'(x)}{x} + \left(1 - \frac{1}{x^2}\right) F(x) = 0, \quad (19)$$

where now the primes denote differentiation with respect to x . Equation (19) is solved by the ordinary Bessel functions of order 1, namely, $J_1(x)$ and $Y_1(x)$. The solutions for the vector potentials inside and outside the solenoid now take the form

$$\begin{aligned} \mathbf{A}_{\text{in}} &= A_1 J_1(\omega\rho) e^{i\omega t} \hat{\phi} \quad \text{for } \rho < R, \\ \mathbf{A}_{\text{out}} &= [C_1 J_1(\omega\rho) + D_1 Y_1(\omega\rho)] e^{i\omega t} \hat{\phi} \quad \text{for } \rho \geq R, \end{aligned} \quad (20)$$

where A_1 , C_1 , D_1 are constants to be determined from boundary conditions. Note that $\mathbf{A}_{\text{in}}$ only uses $J_1(\omega\rho)$ since $Y_1(\omega\rho)$ diverges at $\rho = 0$. From the vector potential in (20) one can calculate the $\mathbf{E}$ and $\mathbf{B}$ fields. Using these new $\mathbf{E}$ and $\mathbf{B}$ fields and repeating the arguments leading to the results in (10) or (11), it is straightforward to verify that the electric and magnetic contributions to the *time dependent* parts of the Aharonov-Bohm phase shift still cancel. Different time dependences for the flux will lead to different forms for $F(\rho)$ but the linear dependence considered previously in (9) and the sinusoidal dependence considered in this section are the most interesting cases from a theoretical point of view and from an experimental point of view. The linear case is interesting since it most clearly illustrates the cancellation of the time dependent part of the Aharonov-Bohm phase shift with very little approximation. Also for a fixed time interval one can arrange to have a linearly increasing flux. The sinusoidal case is probably the easiest situation to set up experimentally and it would most dramatically demonstrate the effect predicted in this work—if the above analysis is correct the Aharonov-Bohm interference pattern should not (contrary to earlier expectations) shift in time to the frequency of the sinusoidally varying flux.

One final point to make in this case of a sinusoidally varying flux is that now there is a non-zero magnetic field outside the solenoid. In the previous case using $\mathbf{A}_{\text{out}}$ from (6) one got $\mathbf{B}_{\text{out}} = \nabla \times \mathbf{A}_{\text{out}} = 0$. However, using $\mathbf{A}_{\text{out}}$ from (20) gives $\mathbf{B}_{\text{out}} = \nabla \times \mathbf{A}_{\text{out}} \neq 0$. Thus, for a sinusoidally varying field there are both $\mathbf{E}$ and $\mathbf{B}$ fields outside the solenoid which makes this conceptually different from the usual time independent Aharonov-Bohm effect. However, as mentioned above the time dependent parts of the electric and magnetic contributions from (10) or (11) still cancel.

V. MULTIVALUED MONOPOLE POTENTIAL

There is another situation where one encounters non-single valued potentials and gauge transformation functions—the potentials for a magnetic monopole. The following 3-vector potential (now using spherical polar coordinates r , θ , ϕ , rather than the cylindrical coordinates, ρ , ϕ , z used in Secs. III and IV)

$$\mathbf{A}_{\text{monopole}} = \frac{g(1 - \cos \theta)}{r \sin \theta} \hat{\phi}, \quad (21)$$

yields a monopole magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = g\hat{\mathbf{r}}/r^2$. The vector potential in (21) is single valued, but it has the usual Dirac string singularity pathology along the negative z -axis, i.e., $\theta = \pi$. One can also obtain a magnetic monopole field from $\mathbf{A}_{\text{monopole}} = -\frac{g(1 + \cos \theta)}{r \sin \theta} \hat{\phi}$ which has a Dirac string singularity along the positive z -axis, i.e., $\theta = 0$. These two forms of the monopole 3-vector potential are related by the gauge transformation $\mathbf{A} \rightarrow \mathbf{A} - \nabla \chi$ with $\chi = 2g\phi$. In this case the gauge function

χ is non-single valued but the two forms of the gauge potential $\mathbf{A}_{\text{monopole}}$ are single valued. Thus, this is not exactly like the time-dependent AB effect of Secs. III and IV.

It is easy to see that one can also get a magnetic monopole field from the following, alternative 3-vector potential^{11,12}

$$\mathbf{A}_{\text{monopole}} = -\frac{g\varphi \sin \theta}{r} \hat{\theta}, \quad (22)$$

which does not have Dirac string singularity of (21) but it is non-single valued due to the φ dependence of A_θ . The two vector potentials in (21) and (22) are related by a gauge transformation of the form

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi; \quad \chi = -g(1 - \cos \theta)\varphi. \quad (23)$$

Here we see that both the gauge transformation function χ in (23) and the 3-vector gauge potential (22) are non-single valued, which then is similar to the time-dependent AB effect of Secs. III and IV. In this work, we just point out the similarity between the time-dependent AB effect and magnetic monopoles. We will return to a detailed analysis of the monopole case in future work.

VI. SUMMARY

In this paper, we have investigated the time-dependent AB effect and related issues connected with Stokes' theorem and gauge symmetry. We found that the 4-dimensional Stokes' theorem as given by (2) with $\omega = A$ and $d\omega = dA = F$ is gauge dependent since $\oint_{\partial c} A$ is gauge dependent. This comes about due to the non-single valued character of scalar potential ϕ from (14) and the non-single valued character of the gauge function χ from (13). This non-single valued character of ϕ and χ are the result of the topology of the Aharonov-Bohm set-up being non-simply connected.⁷ Since the quantity $\oint_{\partial c} A$ is gauge dependent the equality of the left- and right-hand sides of Stokes' theorem given in (2) will only be true for a specific gauge. In terms of the scalar and 3-vector potentials given in (14) the gauge where the left-hand side and right-hand side of Stokes' theorem are equal is $K = 1/2$. This analysis can be tested experimentally by performing the time-dependent Aharonov-Bohm experiment. If one finds that $\delta\alpha_{AB}$ does not inherit the time-variation of the magnetic flux then the above analysis is correct; if $\delta\alpha_{AB}$ inherits the time dependence of the magnetic field then the above analysis is not correct.

We concluded by showing that some of these issues of non-single valuedness of the gauge potentials and gauge transformation function also appear in the analysis of magnetic monopoles. We will return to a more detailed examination of this in future work.

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