Block-diagonal characterization of locally finite simple groups of p -type

Stefaan Delcroix

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Abstract. In this paper, we prove the following characterization of LFS-groups of p -type:

Let G be a locally finite group and $\mathcal{S}$ be a local system for G such that $\mathcal{S} \subseteq \mathcal{P}$ and for all $S \in \mathcal{S}$:

(i) $\{T \in \mathcal{S}(S) \mid V_T^j \text{ is block-diagonal for } S \text{ for } j = 1, \dots, n_T\}$ is a local system for G .

(ii) there exists $F_S \in \mathcal{S}(S)$ such that

(a) $V_{F_S}^j$ is block-diagonal for S for $j = 1, \dots, n_{F_S}$,

(b) $[V_{F_S}^j, S_i^*] \neq 0$ for $i = 1, \dots, n_S$ and $j = 1, \dots, n_{F_S}$,

(c) for $j = 1, \dots, n_{F_S}$,

$$\sum_{i=1}^{n_S} n_{SF_S}^{ij} \dim_{K_{F_S}^j} \langle K_S^i, K_{F_S}^j \rangle \geq 2$$

where $n_{SF_S}^{ij}$ is the number of composition factors C for $\hat{S}$ on $V_{F_S}^j$ with $[C, S_i^*] \neq 0$.

Then $G/\text{LSol}(G)$ is an LFS-group of p -type and $[G, \text{LSol}(G)] \leq O_p(G)$.

We use this theorem to construct a general family of LFS-groups of p -type.

1 Introduction, definitions and notations

The locally finite simple groups have been divided into four types: finitary, 1-type, p -type and ∞ -type (see [3, 5]). The groups of finitary type are those in which elements exhibit ‘no growth’ (for an appropriate definition) and have been classified. In the groups of ∞ -type, all the elements exhibit ‘unrestricted growth’. The groups of 1-type and p -type sit in between: the elements exhibit ‘linear growth’. These groups (1-type and p -type) are the groups of diagonal type because subgroups are embedded in each other in a specific way (‘diagonally’). In the groups of 1-type, these subgroups involve alternating groups while in the groups of p -type, they involve classical groups in characteristic p .

In [1, Section 4.15], the author proved that every group of p -type admits a Kegel cover whose members modulo normal p -subgroups are direct products of classical groups in characteristic p and the embeddings are diagonal. In this paper,

we prove the converse: a locally finite group with a subgroup cover as described above must be simple of p -type modulo a locally solvable normal subgroup. We will use this converse to construct a general family of groups of p -type.

We finish this introduction with some definitions and notations.

Let G be a group. Then G is *locally finite* if every finite subset of G generates a finite subgroup of G . The set $\mathcal{L} = \{H_i \mid i \in I\}$ is a *local system* for G if H_i is a finite subgroup of G for all $i \in I$ such that $\bigcup_{i \in I} H_i = G$ and for all $i, j \in I$, there exists $k \in I$ with $H_i, H_j \leq H_k$. We say that G is an *LFS-group* if G is an infinite, locally finite, simple group. The set $\mathcal{K} = \{(H_i, M_i) \mid i \in I\}$ is a *Kegel cover* for G if H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i for all $i \in I$ such that for every finite subgroup F of G , there exists $i \in I$ with $F \leq H_i$ and $F \cap M_i = 1$. The groups $\{H_i/M_i \mid i \in I\}$ are called the *factors* of the Kegel cover. Note that by [4, p. 113], every LFS-group has a Kegel cover.

Let G be an LFS-group. Then G is *finitary* if there exists a field K and a faithful KG -module V such that $[V, g] := \{v^g - v \mid v \in V\}$ is finite-dimensional for all $g \in G$. We say that G is of p -type for some prime p if G is not finitary and every Kegel cover for G contains at least one factor that is isomorphic to a classical group defined over a field of characteristic p .

If g acts on a K -vector space V , then we put

$$\deg_V(g) = \dim_K[V, g] \quad \text{and} \quad \text{pdeg}_V(g) = \min\{\deg_V(\lambda g) \mid \lambda \in K\}.$$

If G is a finite group, we denote by G^∞ the largest perfect subgroup of G . We refer to G^∞ as the *perfect part* of G .

If S is a finite perfect group, we denote by $\hat{S}$ the universal central extension of S and by $\varphi_S : \hat{S} \rightarrow S$ a fixed epimorphism (note that $\text{Ker } \varphi_S \leq Z(\hat{S})$).

If G is a group, $S \leq G$ and $\mathcal{L}$ is a collection of subgroups of G , then we put $\mathcal{L}(S) = \{L \in \mathcal{L} \mid S \leq L\}$.

For the rest of this paper, let p be a fixed prime. We denote by $\mathcal{P}$ the class of all finite perfect groups S such that $S/O_p(S) = S^1 \dots S^{n_S}$ where

- (1) S^i is a component of $S/O_p(S)$ for $i = 1, \dots, n_S$,
- (2) $S^i/Z(S^i)$ is isomorphic to a classical group defined over a field of characteristic p for $i = 1, \dots, n_S$.

Fix $1 \leq i \leq n_S$. We put $PG_S^i = S^i/Z(S^i)$, which is a classical group over the field K_S^i and the K_S^i -vector space V_S^i , so

$$PG_S^i \in \{\text{PSL}_{K_S^i}(V_S^i), \text{PSp}_{K_S^i}(V_S^i), \text{PSU}_{K_S^i}(V_S^i), \text{P}\Omega_{K_S^i}(V_S^i)\}.$$

We denote the linear classical group by G_S^i , so

$$G_S^i \in \{\text{SL}_{K_S^i}(V_S^i), \text{Sp}_{K_S^i}(V_S^i), \text{SU}_{K_S^i}(V_S^i), \Omega_{K_S^i}(V_S^i)\}.$$

Let S_i be the perfect part of the preimage in S of S^i and let S_i^* be the perfect part of $S_i^{\varphi_S^{-1}}$. Put

$$\pi_S^i : G_S^i \rightarrow PG_S^i : x \rightarrow xZ(G_S^i)$$

and

$$\alpha_S^i : S \rightarrow PG_S^i : s_1 \dots s_{n_S} \rightarrow s_i \text{Sol}(S_i)$$

where $s_j \in S_j$ for $j = 1, \dots, n_S$. Then there exists a unique epimorphism

$$\beta_S^i : \hat{S} \rightarrow G_S^i$$

such that $\beta_S^i \pi_S^i = \varphi_S \alpha_S^i$.

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\varphi_S} & S & \xrightarrow{\alpha_S^i} & PG_S^i \\ & \searrow \beta_S^i & & & \uparrow \pi_S^i \\ & & & & G_S^i \end{array}$$

Let K be a field and V be a finite-dimensional K -vector space. We denote the dual of V by V^* . For $\alpha \in \text{Aut}(K)$, we define a new scalar multiplication $\cdot_\alpha$ on V by putting $v \cdot_\alpha k = vk^\alpha$ for all $v \in V$ and all $k \in K$; we will write V^α to show that we are working with the scalar multiplication $\cdot_\alpha$.

We fix an algebraic closure $\overline{\text{GF}(p)}$ of $\text{GF}(p)$ (in general, $\text{GF}(q)$ is the Galois-field with q elements). For any finite field K with $\text{char } K = p$, we fix an embedding from K into $\overline{\text{GF}(p)}$ (so we view K as a subfield of $\overline{\text{GF}(p)}$). If K and L are finite fields, we will denote by $\langle K, L \rangle$ the subfield of $\overline{\text{GF}(p)}$ generated by K and L .

Definition 1.1. Let $S \in \mathcal{P}$, let K be a finite field with $\text{char } K = p$, let V be a finite-dimensional K -vector space and let $\theta : S \rightarrow \text{PSL}_K(V)$ be a homomorphism. Put $\pi : \text{SL}_K(V) \rightarrow \text{PSL}_K(V)$, $x \rightarrow xZ(\text{SL}_K(V))$. Then there exists a unique epimorphism $\hat{\theta} : \hat{S} \rightarrow \text{SL}_K(V)$ such that $\hat{\theta}\pi = \varphi_S\theta$. We say that V is *block-diagonal* for S if for every non-trivial K -composition factor C for S on V (see the diagrams below)

- (i) there exists exactly one $i \in \{1, \dots, n_S\}$ such that S_i is not trivial on C (or equivalent, $[C, S_i^*] \neq 0$).
- (ii) for this i , there exists $\sigma \in \text{Aut}(K_S^i)$ such that

$$C \cong (V_S^i)^\sigma \otimes_{K_S^i} L \quad \text{or} \quad C \cong (V_S^i)^{*,\sigma} \otimes_{K_S^i} L$$

as $K\hat{S}$ -modules where $L = \langle K, K_S^i \rangle$.

$$\begin{array}{ccc}
 \hat{S} & \xrightarrow{\varphi_S} S & \xrightarrow{\alpha_S^i} PG_S^i \\
 & \searrow \beta_S^i & \uparrow \pi_S^i \\
 & & G_S^i
 \end{array}
 \qquad
 \begin{array}{ccc}
 \hat{S} & \xrightarrow{\varphi_S} S & \xrightarrow{\theta} \mathrm{PSL}_K(V) \\
 & \searrow \hat{\theta} & \uparrow \pi \\
 & & \mathrm{SL}_K(V)
 \end{array}$$

In [1, Section 4.15], the author proved that every LFS-group of p -type has a Kegel cover $\mathcal{K}$ such that $S \in \mathcal{P}$ for all $(S, M) \in \mathcal{K}$ and V_T^i is block-diagonal for S for all $(S, M), (T, N) \in \mathcal{K}$ with $S \leq T$ and all $1 \leq i \leq n_T$. In this paper, we prove more or less the converse.

2 Block-diagonal characterization of LFS-groups of p -type

In [5, Section 3.3], U. Meierfrankenfeld proved that if G is an LFS-group of p -type, then G has a Kegel cover $\mathcal{K}$ such that $H \in \mathcal{P}$ for all $(H, M) \in \mathcal{K}$. The converse however is not true. There exist non-finitary LFS-groups with such a Kegel cover that are not of p -type; P. Hall's universal group is an example. Block-diagonality however is something typical for groups of p -type.

In this section, we prove the following theorem:

Theorem 2.1. *Let G be a locally finite group and $\mathcal{S}$ be a local system for G such that $\mathcal{S} \subseteq \mathcal{P}$ and for all $S \in \mathcal{S}$:*

- (i) *$\{T \in \mathcal{S}(S) \mid V_T^j \text{ is block-diagonal for } S \text{ for } j = 1, \dots, n_T\}$ is a local system for G .*
- (ii) *there exists $F_S \in \mathcal{S}(S)$ such that*
 - (a) *$V_{F_S}^j$ is block-diagonal for S for $j = 1, \dots, n_{F_S}$,*
 - (b) *$[V_{F_S}^j, S_i^*] \neq 0$ for $i = 1, \dots, n_S$ and $j = 1, \dots, n_{F_S}$.*
 - (c) *for $j = 1, \dots, n_{F_S}$,*

$$\sum_{i=1}^{n_S} n_{SF_S}^{ij} \dim_{K_{F_S}^j} \langle K_S^i, K_{F_S}^j \rangle \geq 2$$

where for $1 \leq i \leq n_S$ and $1 \leq j \leq n_{F_S}$, we denote by $n_{SF_S}^{ij}$ the number of composition factors C for $\hat{S}$ on $V_{F_S}^j$ with $[C, S_i^*] \neq 0$; note that $n_{SF_S}^{ij} \geq 1$ by (b) since S_i^* is perfect.

Then $G/\mathrm{LSol}(G)$ is an LFS-group of p -type and $[G, \mathrm{LSol}(G)] \leq O_p(G)$.

Note that conditions (i) and (ii)(a)–(b) imply that G is a direct limit of subgroups in which classical groups act non-trivially and block-diagonally. Further,

condition (ii)(c) is critical as it forces $G/\text{LSol}(G)$ to be of p -type. Without this condition, $G/\text{LSol}(G)$ would be finitary.

Before proving this theorem, we provide an example on how this theorem can be used to construct LFS-groups of p -type. The next section will include a general construction of LFS-groups of p -type.

Let K be a finite field with $\text{char } K = p$ and V_0 be a finite-dimensional K -vector space such that $\text{SL}_K(V_0)$ is perfect. Put $n = \dim_K V_0$. For $i \geq 1$, define

$$V_i = V_{i-1} \oplus V_{i-1}.$$

Fix a K -basis β for V_0 . Then β induces in a natural way a K -basis for V_i for $i \geq 1$. All matrix representations are with respect to these bases. For $i \geq 1$, put

$$G_i = \left\{ \begin{bmatrix} A & 0 \\ X & B \end{bmatrix} \mid A, B \in \text{SL}_K(V_{i-1}), X \in K^{2^{i-1}n \times 2^{i-1}n} \right\} \leq \text{SL}_K(V_i).$$

For $g \in G_i$, put

$$g = \begin{bmatrix} A_{g,1} & 0 \\ X_g & A_{g,2} \end{bmatrix}.$$

For $i \geq 1$, consider the embedding

$$\varphi_i : G_i \rightarrow G_{i+1} : \begin{bmatrix} A & 0 \\ X & B \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 & 0 & 0 \\ X & B & 0 & 0 \\ 0 & 0 & A & 0 \\ X & 0 & X & B \end{bmatrix}.$$

Let G be the direct limit of the G_i . We view G_i as a subgroup of G for $i \geq 1$ (and so $G_i \leq G_{i+1}$). Then G is an infinite locally finite group and $\{G_i \mid i \geq 1\}$ is a local system for G .

Fix $i \geq 1$. Put $S = G_i$.

First, we show that $S \in \mathcal{P}$. Put $m = 2^{i-1}n \times 2^{i-1}n$. Put

$$S_1 = \{s \in S \mid A_{s,2} = I\} \quad \text{and} \quad S_2 = \{s \in S \mid A_{s,1} = I\}.$$

Pick $A \in \text{SL}_K(V_{i-1})$ and $X \in K^{m \times m}$. Pick a matrix $B \in \text{SL}_K(V_{i-1})$ such that $\det(B - I) \neq 0$ (we can do this by choosing B such that its characteristic polynomial is $(-1)^m(X^m - X^{m-1} + (-1)^m)$). Since $\text{SL}_K(V_{i-1}) \cong \{s \in S_1 \mid X_s = 0\}$ and $\text{SL}_K(V_{i-1})$ is perfect, there exists $s \in S'_1$ with $A_{s,1} = A$ and $X_s = 0$. Then

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \left[\begin{bmatrix} B^{-1} & 0 \\ X(B - I)^{-1} & I \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \right] = \begin{bmatrix} A & 0 \\ X & I \end{bmatrix} \in S'_1.$$

So $S'_1 = S_1$. Similarly, $S'_2 = S_2$. Since

$$\begin{bmatrix} A & 0 \\ X & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},$$

we see that $S = S_1 S_2$. Hence $S = S'$. Note that

$$O_p(S) = \{s \in S \mid A_{s,1} = A_{s,2} = I\}$$

and

$$S/O_p(S) \cong \mathrm{SL}_K(V_{i-1}) \times \mathrm{SL}_K(V_{i-1}).$$

So $S \in \mathcal{P}$ and $n_S = 2$.

Let $t > 0$. Put $T = G_{i+t}$. We show that V_T^k is block-diagonal for S for $k = 1, 2$. For $j = 1, 2$, put $\gamma_S^j : S \rightarrow G_S^j$, $s \mapsto A_{s,j}$. Then $G_S^j = \mathrm{SL}_K(V_{i-1})$, $\alpha_S^j = \gamma_S^j \pi_S^j$ and $\beta_S^j = \varphi_S \gamma_S^j$ for $j = 1, 2 = n_S$. We have similar notations and results for T (see the diagrams below).

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\varphi_S} & S & \xrightarrow{\alpha_S^j} & PG_S^j \\ & \searrow \beta_S^j & \searrow \gamma_S^j & \nearrow \pi_S^j & \uparrow \\ & & G_S^j & & \end{array} \quad \begin{array}{ccccc} \hat{T} & \xrightarrow{\varphi_T} & T & \xrightarrow{\alpha_T^k} & PG_T^k \\ & \searrow \beta_T^k & \searrow \gamma_T^k & \nearrow \pi_T^k & \uparrow \\ & & G_T^k & & \end{array}$$

Pick $k \in \{1, 2\}$. Using the notations of block-diagonality (with ' $\theta = \alpha_T^k$ '; see the diagram below), we get that

$$\widehat{\alpha_T^k} = \varphi_S \gamma_T^k.$$

Note that S has exactly 2^t composition factors on $V_T^k = V_{i+t-1}$ and all of them are non-trivial. The set S_1 is non-trivial and S_2 is trivial on one half of these composition factors while S_1 is trivial and S_2 is non-trivial on the other half. Let C be a composition factor for S on V_T^k . Then there exists $j \in \{1, 2\}$ such that $[C, S_l] = 0$ for $l \in \{1, 2\}$ with $l \neq j$ and $C \cong V_S^j = V_{i-1}$ as KS -modules. Since

$$\varphi_S \gamma_S^j = \beta_S^j \quad \text{and} \quad \varphi_S \gamma_T^k = \widehat{\alpha_T^k},$$

V_T^k is block-diagonal for S .

$$\begin{array}{ccccccc} \hat{S} & \xrightarrow{\varphi_S} & S & \xrightarrow{\alpha_T^k} & PG_T^k & \xrightarrow{\tau_C} & \mathrm{PSL}_{K_T^k}(C) \\ \downarrow \beta_S^j & \searrow \gamma_S^j & \searrow \widehat{\alpha_T^k} & \nearrow \gamma_T^k & \nearrow \pi_T^k & & \uparrow \pi_C \\ & & G_S^j & & G_T^k & \xrightarrow{\tau_C} & \mathrm{SL}_{K_T^k}(C) \end{array}$$

It follows from Theorem 2.1 that $G/\text{LSol}(G)$ is an LFS-group of p -type. Note that $\text{LSol}(G) = Z(G)$. Indeed, for $i \geq 1$, we have that

$$\begin{aligned}\text{Sol}(G_i) &= \{g \in G_i \mid A_{g,1}, A_{g,2} \in Z(\text{SL}_K(V_{i-1}))\}, \\ Z(G_i) &= \{g \in G_i \mid X_g = 0, A_{g,1} = A_{g,2} \in Z(\text{SL}_K(V_{i-1}))\}, \\ Z(G_i) &\leq Z(G_{i+1}).\end{aligned}$$

Hence $Z(G) = \bigcup_{i=1}^{\infty} Z(G_i)$. Pick $g \in \text{LSol}(G)$ and $i \geq 1$ with $g \in G_i$. Then $g \in \text{Sol}(G_i)$ and $g \in \text{Sol}(G_{i+1})$. Hence, as element of G_i , we get that $X_g = 0$ and $A_{g,1} = A_{g,2} \in Z(\text{SL}_K(V_{i-1}))$. So $g \in Z(G_i) \leq Z(G)$. Thus $\text{LSol}(G) = Z(G)$.

We now proceed with the proof of Theorem 2.1. First, we need a couple of lemmas.

Lemma 2.2. *Let K be a finite field with $\text{char } K = p$, V be a finite-dimensional K -vector space, $G \in \{\text{SL}_K(V), \text{Sp}_K(V), \text{SU}_K(V), \Omega_K(V)\}$, P be a p -subgroup of G , $k \geq 1$ and $x \in P$ with $\text{pdeg}_V(x) > 2^{k+1}|P|^2 + f(|P|)$ (where f is the function of [5, Section 2.2]). Then there exists a p -subgroup P^* of G such that $P \leq P^*$ and $\text{der}(\langle x^{P^*} \rangle) \geq k$.*

Proof. The case ' $G = \text{SL}_K(V)$ ' follows from [5, Section 5.2]. So assume that $G \neq \text{SL}_K(V)$. By [5, Section 2.3 (c)] (with ' $G = P$ ', ' $m = |P|$ ', ' $H = \{x\}$ ' and ' $t = |P|2^k$ '), there exists a P -invariant singular subspace U of V with

$$\text{pdeg}_U(x) \geq |P|2^k.$$

With respect to a certain basis of V , every $g \in N_G(U)$ is of the form

$$\begin{bmatrix} A_g & 0 & 0 \\ * & B_g & 0 \\ * & * & A_g^{-\sigma t} \end{bmatrix}$$

where $\sigma \in \text{Aut}(K)$ with $\sigma^2 = 1$, $A_g \in \text{SL}_K(U)$ and $B_g \in \tilde{G}$ ($\tilde{G}$ is the corresponding element in $\{\text{Sp}_K(U^\perp/U), \text{SU}_K(U^\perp/U), \Omega_K(U^\perp/U)\}$). Put

$$\begin{aligned}\pi_1 : N_G(U) &\rightarrow \text{SL}_K(U), & g &\rightarrow A_g, \\ \pi_2 : N_G(U) &\rightarrow \tilde{G}, & g &\rightarrow B_g, \\ \pi : N_G(U) &\rightarrow \text{SL}_K(U) \times \tilde{G}, & g &\rightarrow (A_g, B_g).\end{aligned}$$

Note that $\text{pdeg}_U(A_x) \geq \text{pdeg}_U(x) \geq |P|2^k \geq |P|^{\pi_1}|2^k|$. By [5, Section 5.2], there exists a p -subgroup P_1 of $\text{SL}_K(U)$ such that $P^{\pi_1} \leq P_1$ and $\text{der}(\langle (x^{\pi_1})^{P_1} \rangle) \geq k$.

Put $P^* = (P_1 \times P^{\pi_2})^{\pi^{-1}}$. Then P^* is a p -group since $P^*/\text{Ker } \pi$ and $\text{Ker } \pi$ are p -groups and $P \leq P^*$ since $P^\pi \leq P^{\pi_1} \times P^{\pi_2} \leq P_1 \times P^{\pi_2}$. Finally,

$$\langle x^{P^*} \rangle^{\pi_1} = \langle (x^{\pi_1})^{P_1} \rangle$$

and so $\text{der}(\langle x^{P^*} \rangle) \geq \text{der}(\langle x^{P^*} \rangle^{\pi_1}) \geq k$. $\square$

Corollary 2.3. *Lemma 2.2 remains true if*

$$G \in \{\text{PSL}_K(V), \text{PSp}_K(V), \text{PSU}_K(V), \text{P}\Omega_K(V)\}.$$

Proof. Pick $H \in \{\text{SL}_K(V), \text{Sp}_K(V), \text{SU}_K(V), \Omega_K(V)\}$ with $H/Z(H) = G$. Let Q be a p -subgroup of H with $QZ(H)/Z(H) = P$ and $y \in Q$ with $yZ(H) = x$. Note that $Q \cap Z(H) = 1$ and so $|Q| = |P|$. By Lemma 2.2, there exists a p -subgroup Q^* of H such that $Q \leq Q^*$ and $\text{der}(\langle y^{Q^*} \rangle) \geq k$. Put

$$P^* = Q^*Z(H)/Z(H).$$

Then P^* is a p -subgroup of G containing P and $\text{der}(\langle x^{P^*} \rangle) \geq k$ since

$$\langle y^{Q^*} \rangle \cong \langle y^{Q^*} \rangle Z(H)/Z(H) = \langle (yZ(H))^{Q^*Z(H)/Z(H)} \rangle = \langle x^{P^*} \rangle. \quad \square$$

Lemma 2.4. *Let F, K, L be finite fields with $F, K \subseteq L$, let V be a finite-dimensional K -vector space, let $\alpha \in \text{GL}_K(V)$ and $\sigma \in \text{Aut}(K)$. Then*

$$\deg_{V^\sigma \otimes_K L}(\alpha) = \deg_{V^{*,\sigma} \otimes_K L}(\alpha) = \deg_V(\alpha)[L : F]$$

where we view $V^\sigma \otimes_K L$ and $V^{*,\sigma} \otimes_K L$ as F -vector spaces.

Proof. Clearly, $\deg_V(\alpha) = \deg_{V^\sigma}(\alpha)$. Let A be the matrix representation of α with respect to a certain basis of V . Then A^{-t} is the matrix representation of α with respect to a certain basis of V^* . Put $n = \dim_K V$. Then for all $x \in K^n$, we have that $xA = x$ if and only if $xA^{-1} = x$. Hence the matrices $A - I$, $A^{-1} - I$ and $A^{-t} - I$ have the same rank. So $\deg_V(\alpha) = \deg_{V^*}(\alpha)$. Thus we only need to prove the case ' $V \otimes_K L$ '. Put $W = V \otimes_K L$. Then

$$\text{Ker}(\alpha - 1_V) \otimes_K L \subseteq \text{Ker}(\alpha \otimes_K 1_L - 1_W)$$

and

$$\text{Im}(\alpha - 1_V) \otimes_K L \subseteq \text{Im}(\alpha \otimes_K 1_L - 1_W).$$

Comparing the dimensions over L , we see that we have equality. Hence

$$\begin{aligned} \deg_W(\alpha) &= \dim_F(\text{Im}(\alpha \otimes_K 1_L - 1_W)) \\ &= \dim_F(\text{Im}(\alpha - 1_V) \otimes_K L) \\ &= \dim_K(\text{Im}(\alpha - 1_V))[L : F] \\ &= \deg_V(\alpha)[L : F] \end{aligned}$$

which proves the lemma. $\square$

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. First, we show that $G/\text{LSol}(G)$ is simple and that

$$[G, \text{LSol}(G)] \leq O_p(G).$$

Pick $x \in G \setminus \text{LSol}(G)$. Suppose that $x \in \text{Sol}(R)$ for all $R \in \mathcal{S}(x)$. Let F be a finite subgroup of $\langle x^G \rangle$. Pick $R \in \mathcal{S}(x)$ such that $F \leq \langle x^R \rangle$. Then $F \leq \text{Sol}(R)$. So $\langle x^G \rangle$ is locally solvable and $\langle x^G \rangle \leq \text{LSol}(G)$, a contradiction. Hence there exists $R \in \mathcal{S}(x)$ with $x \notin \text{Sol}(R)$. Pick $S \in \mathcal{S}(R)$. Then $x \notin \text{Sol}(S)$. Put $T = F_S$. Put $x = s_1 \dots s_{n_S} = t_1 \dots t_{n_T}$ where $s_i \in S_i$ for $i = 1, \dots, n_S$ and $t_j \in T_j$ for $j = 1, \dots, n_T$. Since $x \notin \text{Sol}(S)$, there exists $1 \leq i \leq n_S$ with $s_i \notin \text{Sol}(S_i)$. Pick $1 \leq j \leq n_T$. Since $[V_T^j, S_i^*] \neq 0$ and S_i^* is perfect, there exists a composition factor C for $\hat{S}$ on V_T^j with

$$[C, S_i^*] \neq 0.$$

By block-diagonality, there exists $\sigma \in \text{Aut}(K_S^i)$ such that

$$C \cong (V_S^i)^\sigma \otimes_{K_S^i} \langle K_S^i, K_T^j \rangle$$

or

$$C \cong (V_S^i)^{*,\sigma} \otimes_{K_S^i} \langle K_S^i, K_T^j \rangle$$

as $K_T^j \hat{S}$ -modules. Pick $y \in \hat{S}$ with $y^{\varphi_S} = x$. We use the notations of the definition of block-diagonality with ' $\theta = \alpha_T^j$ ' (see the diagram below). Since $s_i \notin \text{Sol}(S_i)$, $1 \neq x^{\alpha_S^i} = y^{\varphi_S \alpha_S^i} = y^{\beta_S^i \pi_S^i}$ and so y does not act like a scalar on V_S^i . Hence y does not act like a scalar on C , nor on V_T^j . So

$$1 \neq y^{\hat{\alpha}_T^j \pi_T^j} = y^{\varphi_S \alpha_T^j} = x^{\alpha_T^j}.$$

Hence $t_j \notin \text{Sol}(T_j)$.

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\varphi_S} & S \xrightarrow{\alpha_S^i} P G_S^i \\ & \searrow \beta_S^i & \uparrow \pi_S^i \\ & & G_S^i \end{array} \qquad \begin{array}{ccc} \hat{S} & \xrightarrow{\varphi_S} & S \xrightarrow{\alpha_T^j} P G_T^j \\ & \searrow \hat{\alpha}_T^j & \uparrow \pi_T^j \\ & & G_T^j \end{array}$$

By [1, Section 1.6 and 1.7 (b)],

$$\langle x^T \rangle \text{Sol}(T) / \text{Sol}(T) = P G_T^1 \times \dots \times P G_T^{n_T}$$

and so $\langle x^T \rangle \text{Sol}(T) = T$. Taking perfect parts, we get that

$$T = T^\infty = \langle x^T \rangle^\infty \leq \langle x^T \rangle$$

and so $\langle x^T \rangle = T$. Hence

$$S \leq \langle x^G \rangle \quad \text{for all } S \in \mathcal{S}(R).$$

Thus $\langle x^G \rangle = G$ and $G/\text{LSol}(G)$ is simple.

Let F be a finite subgroup of G and H be a finite subgroup of $\text{LSol}(G)$. Pick $S \in \mathcal{S}(\langle F, H \rangle)$. Then $H \leq S \cap \text{LSol}(G) \leq \text{Sol}(S)$ and so

$$[F, H] \leq [S, \text{Sol}(S)] \leq O_p(S).$$

Hence $[G, \text{LSol}(G)]$ is a p -group and so $[G, \text{LSol}(G)] \leq O_p(G)$.

Next, we show that $G/\text{LSol}(G)$ is infinite and not finitary by constructing a non-solvable p -subgroup. Condition (ii) (c) is crucial here.

Pick $x \in G$ such that $|x|$ is a power of p and $x \notin \text{LSol}(G)$. We prove two claims needed to construct this non-solvable p -subgroup.

Claim 1. *For all $S \in \mathcal{S}(x)$ and $k \geq 1$, there exists $T_k \in \mathcal{S}(S)$ with*

$$\text{pdeg}_{V_{T_k}^j}(x) \geq 2^{k-1} \quad \text{for } j = 1, \dots, n_{T_k}.$$

Pick $S \in \mathcal{S}(x)$. We induct on k . Pick $T_0 \in \mathcal{S}(S)$ with $x \notin \text{Sol}(T_0)$. Suppose now that $k \geq 1$. Put $T_k = F_{T_{k-1}}$. Pick $y \in \widehat{T_{k-1}}$ such that $|y|$ is a power of p and $y^{\varphi_{T_{k-1}}} = x$. For $i = 1, \dots, n_{T_{k-1}}$,

$$y^{\beta_{T_{k-1}}^i} \pi_{T_{k-1}}^i = y^{\varphi_{T_{k-1}}} \alpha_{T_{k-1}}^i = x^{\alpha_{T_{k-1}}^i}$$

and so

$$\text{pdeg}_{V_{T_{k-1}}^i}(x) = \text{pdeg}_{V_{T_{k-1}}^i}(y).$$

Similarly,

$$\text{pdeg}_{V_{T_k}^j}(x) = \text{pdeg}_{V_{T_k}^j}(y) \quad \text{for } j = 1, \dots, n_{T_k}.$$

Let $1 \leq j \leq n_{T_k}$. Using the definition of block-diagonality and Lemma 2.4, we get that

$$\text{pdeg}_{V_{T_k}^j}(y) \geq \sum_{i=1}^{n_{T_{k-1}}} n_{T_{k-1}T_k}^{ij} \text{pdeg}_{V_{T_{k-1}}^i}(y) \dim_{K_{T_k}^j} \langle K_{T_{k-1}}^i, K_{T_k}^j \rangle.$$

If $k = 1$, then there exists $1 \leq i \leq n_{T_0}$ with $x \notin \text{Ker } \alpha_{T_0}^i$ since $x \notin \text{Sol}(T_0)$ and so

$$\text{pdeg}_{V_{T_0}^i}(x) \geq 1.$$

Hence

$$\text{pdeg}_{V_{T_1}^j}(y) \geq \sum_{i=1}^{n_{T_0}} n_{T_0T_1}^{ij} \text{pdeg}_{V_{T_0}^i}(y) \dim_{K_{T_1}^j} \langle K_{T_0}^i, K_{T_1}^j \rangle \geq 1.$$

If $k \geq 2$, we can use induction to get that

$$\begin{aligned} \text{pdeg}_{V_{T_k}^j}(y) &\geq \sum_{i=1}^{n_{T_{k-1}}} n_{T_{k-1}T_k}^{ij} \text{pdeg}_{V_{T_{k-1}}^i}(y) \dim_{K_{T_k}^j} \langle K_{T_{k-1}}^i, K_{T_k}^j \rangle \\ &\geq 2^{k-2} \sum_{i=1}^{n_{T_{k-1}}} n_{T_{k-1}T_k}^{ij} \dim_{K_{T_k}^j} \langle K_{T_{k-1}}^i, K_{T_k}^j \rangle \\ &\geq 2^{k-2} \cdot 2 = 2^{k-1}. \end{aligned}$$

This proves Claim 1.

Claim 2. *There exist finite p -subgroups $P_1, P_2, \dots$ of G such that*

$$x \in P_1 \leq P_2 \leq \dots$$

and $\text{der}(\langle x^{P_k} \rangle \text{LSol}(G)/\text{LSol}(G)) \geq k$ for $k = 1, 2, \dots$

We induct on k . For $k = 1$, we put $P_1 = \langle x \rangle$ since $x \notin \text{LSol}(G)$. So suppose $k \geq 2$. Pick $S \in \mathcal{S}(P_{k-1})$. By Claim 1, there exists $T \in \mathcal{S}(S)$ such that

$$\text{pdeg}_{V_T^1}(x) > 2^{k+1}|P_{k-1}|^2 + f(|P_{k-1}|)$$

(f is the function of Lemma 2.2). Note that $|P_{k-1}^{\alpha_T^1}| \leq |P_{k-1}|$. By Corollary 2.3, there exists a p -subgroup P^* of PG_T^1 such that

$$P_{k-1}^{\alpha_T^1} \leq P^* \quad \text{and} \quad \text{der}(\langle (x^{\alpha_T^1})^{P^*} \rangle) \geq k.$$

Note that $O_p(T) \leq \text{Ker } \alpha_T^1$. Hence there exists a p -subgroup P_k of T such that

$$P_{k-1}, O_p(T) \leq P_k \quad \text{and} \quad P_k^{\alpha_T^1} = P^*.$$

Then

$$\langle x^{P_k} \rangle^{\alpha_T^1} = \langle (x^{\alpha_T^1})^{P_k^{\alpha_T^1}} \rangle = \langle (x^{\alpha_T^1})^{P^*} \rangle$$

and so $\text{der}(\langle x^{P_k} \rangle^{\alpha_T^1}) \geq k$. Note that

$$\begin{aligned} M &:= \langle x^{P_k} \rangle \cap \text{LSol}(G) = \langle x^{P_k} \rangle \cap T \cap \text{LSol}(G) \\ &\leq \langle x^{P_k} \rangle \cap \text{Sol}(T) \leq \langle x^{P_k} \rangle \cap \text{Ker } \alpha_T^1 := N \end{aligned}$$

and so

$$\langle x^{P_k} \rangle^{\alpha_T^1} \cong \langle x^{P_k} \rangle / \langle x^{P_k} \rangle \cap \text{Ker } \alpha_T^1 = \langle x^{P_k} \rangle / N \cong (\langle x^{P_k} \rangle / M) / (N / M).$$

Hence

$$\begin{aligned} \text{der}(\langle x^{P_k} \rangle \text{LSol}(G) / \text{LSol}(G)) &= \text{der}(\langle x^{P_k} \rangle / \langle x^{P_k} \rangle \cap \text{LSol}(G)) \\ &\geq \text{der}(\langle x^{P_k} \rangle^{\alpha_T^1}) \geq k. \end{aligned}$$

This proves Claim 2.

Let $P_1, P_2, \dots$ be as in Claim 2. Put $P = \bigcup_{k=1}^{\infty} P_k$. Then P is a p -subgroup of G , $x \in P$ but $\langle x^P \rangle \text{LSol}(G) / \text{LSol}(G)$ is not solvable. So $G / \text{LSol}(G)$ is infinite. By [6, Proposition 1], $G / \text{LSol}(G)$ is not finitary.

Finally, we show that $G / \text{LSol}(G)$ is of p -type. By [2, Section 2.3], there exists an odd prime $q \neq p$ with $2\alpha(q) + 1 < q$ where $\alpha(q)$ is the order of p modulo q . By the first claim, there exists $S \in \mathcal{S}$ with

$$\dim_{K_S^1} V_S^1 > 2\alpha(q)(p-1) + 1.$$

Pick $g \in G_S^1$ as in [2, Section 2.4], so

$$|g| = q, \quad g \notin Z(\text{GL}_{K_S^1}(V_S^1)),$$

the matrix representation B of g with respect to a certain K_S^1 -basis for V_S^1 has all its entries in $\text{GF}(p)$, 1 is a root of the characteristic polynomial $f_{K_S^1}(g)$ of g , $f_{K_S^1}(g)$ has at most $2\alpha(q) + 1$ different roots over $\text{GF}(p)$ and

$$f_{K_S^1}(B) = f_{K_S^1}(B^{-1}).$$

Pick $y \in S_1^*$ such that $|y|$ is a power of q and $y^{\beta_S^1} = g$. Put $x = y^{\varphi_S}$. By [2, Section 2.4(b)],

$$1 \neq g^{\pi_S^1} = y^{\beta_S^1 \pi_S^1} = y^{\varphi_S \alpha_S^1} = x^{\alpha_S^1}.$$

So we have $x \notin \text{Sol}(S) \leq \text{Ker } \alpha_S^1$ and $x \notin \text{LSol}(G)$. Let $\tilde{Q}$ be a finite q -subgroup of $G / \text{LSol}(G)$ with $x \text{LSol}(G) \in \tilde{Q}$. Then $\tilde{Q} = Q \text{LSol}(G) / \text{LSol}(G)$ for some finite q -subgroup Q of G with $x \in Q$. Pick $T \in \mathcal{S}(S, Q)$ such that V_T^j is block-diagonal for S for $j = 1, \dots, n_T$. Let $\varphi : \hat{S} \rightarrow \hat{T}$ be the unique homomorphism with $\varphi \varphi_T = \varphi_S$. Pick $1 \leq j \leq n_T$. We use the notations of the definition of block-diagonality with ' $\theta = \alpha_T^j$ ', (see the diagram below). Note that

$$\varphi \beta_T^j \pi_T^j = \varphi \varphi_T \alpha_T^j = \varphi_S \alpha_T^j = \widehat{\alpha_T^j} \pi_T^j.$$

So $\widehat{\alpha_T^j} = \varphi \beta_T^j$ since $\widehat{\alpha_T^j}$ is unique with this property. Put $\tilde{y} = y^{\varphi \beta_T^j}$. Let $f_{K_T^j}(\tilde{y})$ be the characteristic polynomial of $\tilde{y}$ and $f_{K_S^1}(g)$ the characteristic polynomial of g . Using block-diagonality, [2, Section 2.5] and [2, Section 2.4], we see that

$$f_{K_T^j}(\tilde{y}) = (X - 1)^m f_{K_S^1}(g)^n$$

for some $m, n \geq 0$ and $\tilde{y}^q$ is trivial on every composition factor for $\hat{S}$ on V_T^j . So $\tilde{y}^q = 1$ since $|\tilde{y}|$ is a power of q and $q \neq p$. Note that

$$\tilde{y}^{\pi_T^j} = y^{\varphi\beta_T^j\pi_T^j} = y^{\varphi\varphi_T\alpha_T^j} = y^{\varphi_S\alpha_T^j} = x^{\alpha_T^j}.$$

Hence there exists a q -subgroup Q_* of G_T^j such that $\tilde{y} \in Q_*$ and $\text{Ih} Q_*^{\pi_T^j} = Q_*^{\alpha_T^j}$.

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\varphi_S} S & \xrightarrow{\alpha_T^j} PG_T^j \\ & \searrow \widehat{\alpha_T^j} & \uparrow \pi_T^j \\ & & G_T^j \end{array} \quad \begin{array}{ccc} \hat{S} & \xrightarrow{\varphi} \hat{T} & \xrightarrow{\varphi_T} T & \xrightarrow{\alpha_T^j} PG_T^j \\ & \searrow \beta_T^j & \uparrow \pi_T^j \\ & & G_T^j \end{array}$$

By [2, Section 2.4 (d)(e)], the polynomial $f_{K_T^j}(\tilde{y})$ has at most $2\alpha(q) + 1$ different roots over $\overline{\text{GF}(p)}$. Hence $X^q - 1$ does not divide $f_{K_T^j}(\tilde{y})$ since $X^q - 1$ has q different roots over $\overline{\text{GF}(p)}$ and $2\alpha(q) + 1 < q$. If $\tilde{y} = 1$, then $\langle \tilde{y}^{Q_*} \rangle = 1$ and so abelian. If $|\tilde{y}| = q$, then $\langle \tilde{y}^{Q_*} \rangle$ is abelian by [2, Section 2.6]. Hence

$$\langle \tilde{y}^{Q_*} \rangle^{\pi_T^j} = \langle (\tilde{y}^{\pi_T^j})^{Q_*^{\pi_T^j}} \rangle = \langle (x^{\alpha_T^j})^{Q_*^{\alpha_T^j}} \rangle = \langle x^{Q_*} \rangle^{\alpha_T^j}$$

is abelian. So $\langle x^{Q_*} \rangle^{\alpha_T^j}$ is abelian for $j = 1, \dots, n_T$. Since

$$\bigcap_{j=1}^{n_T} \text{Ker } \alpha_T^j = \text{Sol}(T),$$

we get that $\langle x^{Q_*} \rangle \text{Sol}(T) / \text{Sol}(T)$ is abelian. If $X \subseteq T$, we put

$$\overline{X} = XO_p(T) / O_p(T).$$

Then

$$\langle x^{Q_*} \rangle \text{Sol}(T) / \text{Sol}(T) \cong \langle \overline{x}^{\overline{Q_*}} \rangle \overline{\text{Sol}(T)} / \overline{\text{Sol}(T)} \cong \langle \overline{x}^{\overline{Q_*}} \rangle / \langle \overline{x}^{\overline{Q_*}} \rangle \cap \overline{\text{Sol}(T)}.$$

Note that $\overline{\text{Sol}(T)} = \text{Sol}(\overline{T}) = Z(\overline{T})$ and so $\langle \overline{x}^{\overline{Q_*}} \rangle$ is solvable of length at most two. But

$$\langle \overline{x}^{\overline{Q_*}} \rangle = \overline{\langle x^{Q_*} \rangle} = \langle x^{Q_*} \rangle O_p(T) / O_p(T) \cong \langle x^{Q_*} \rangle / \langle x^{Q_*} \rangle \cap O_p(T) = \langle x^{Q_*} \rangle$$

since $\langle x^{Q_*} \rangle \cap O_p(T) = 1$. Hence $\langle x^{Q_*} \rangle$ is solvable of length at most two and so

$$\langle x^{Q_*} \rangle \text{LSol}(G) / \text{LSol}(G) = \langle (x \text{LSol}(G))^{\overline{Q_*}} \rangle$$

is solvable of length at most two. Since $G/\text{LSol}(G)$ is locally finite, it follows that the group $\langle (x \text{LSol}(G))^{\overline{Q_*}} \rangle$ is solvable for all q -subgroups $\overline{Q_*}$ of $G/\text{LSol}(G)$ with $x \text{LSol}(G) \in \overline{Q_*}$. By [2, Section 3.1 (b)], $G/\text{LSol}(G)$ is of p' -type for some

prime p' . Note that $\{S\text{LSol}(G)/\text{LSol}(G) \mid S \in \mathcal{S}\}$ is a local system for the quotient $G/\text{LSol}(G)$. Moreover, if $S \in \mathcal{S}$, then $S/\text{Sol}(S) \cong PG_S^1 \times \cdots \times PG_S^n$ and so all non-abelian composition factors of S are classical groups in characteristic p . The same is true for $S\text{LSol}(G)/\text{LSol}(G)$ since

$$S\text{LSol}(G)/\text{LSol}(G) \cong S/S \cap \text{LSol}(G)$$

and $S \cap \text{LSol}(G) \leq \text{Sol}(S)$. By [5, Section 2.15(d)], $G/\text{LSol}(G)$ has a Kegel cover all of whose factors are classical groups in characteristic p . It follows now from the definition of p' -type that $p' = p$. $\square$

3 General construction of LFS-groups of p -type

In this section, we give a general construction of LFS-groups of p -type, which will be a special case of Theorem 2.1: $S = G_S^1 \times \cdots \times G_S^{n_S}$ for all $S \in \mathcal{S}$. Note that $\text{LSol}(G) = Z(G)$ in this special case. Indeed, pick $x \in \text{LSol}(G)$ and $g \in G$. Then there exists $S \in \mathcal{S}$ with $x, g \in S$. Hence $x \in S \cap \text{LSol}(G) \leq \text{Sol}(S) = Z(S)$ and so $xg = gx$. Thus $x \in Z(G)$ and $\text{LSol}(G) \leq Z(G)$. Hence $\text{LSol}(G) = Z(G)$.

We start with an example that illustrates the general method. Let K be a finite field with $\text{char } K = p$ and V_n be a K -vector space with $\dim_K V_n \geq 2$ for all $n \geq 1$. Put $V = V_1 \otimes_K V_2 \otimes_K V_3 \otimes_K \cdots$. For $n \geq 1$, put $G_n = \text{SL}_K(V_1 \otimes_K \cdots \otimes_K V_n)$. We can view G_n as a subgroup of $\text{GL}_K(V)$ via the embedding $G_n \rightarrow \text{GL}_K(V)$, $g \rightarrow g \otimes_K 1_{W_n}$ where $W_n = V_{n+1} \otimes_K V_{n+2} \otimes_K \cdots$. Note that the composition factors of G_n on V are isomorphic to $V_1 \otimes_K \cdots \otimes_K V_n$. So we have block-diagonal embeddings. Put $G = \bigcup_{n \geq 1} G_n$. Then $G/Z(G)$ is an LFS-group of p -type.

We can generalize this as follows.

Let I be an infinite set, K_i be a finite subfield of $\overline{\text{GF}(p)}$ and V_i be a finite-dimensional K_i -vector space of dimension at least two for all $i \in I$.

For $\emptyset \neq J \subseteq I$, put $K_J = \langle K_j \mid j \in J \rangle$ (the field generated by the K_j with $j \in J$) and $V_J = \bigotimes_{j \in J} (V_j \otimes_{K_j} K_J)$ where the tensor product is taken over K_j . We put $K = K_I$ and $V = V_I$.

Let $\emptyset \neq J \subseteq L \subseteq I$. We view V_L as $(V_J \otimes_{K_J} K_L) \otimes_{K_L} (V_{L \setminus J} \otimes_{K_{L \setminus J}} K_L)$. Put $\theta_{JL} : \text{GL}_{K_J}(V_J) \rightarrow \text{GL}_{K_L}(V_L)$ by

$$((v \otimes_{K_J} 1_{K_L}) \otimes_{K_L} (w \otimes_{K_{L \setminus J}} 1_{K_L}))^{g^{\theta_{JL}}} = (v^g \otimes_{K_J} 1_{K_L}) \otimes_{K_L} (w \otimes_{K_{L \setminus J}} 1_{K_L})$$

for all $v \in V_J$, $w \in V_{L \setminus J}$ and $g \in \text{GL}_{K_J}(V_J)$. Then θ_{JL} is a monomorphism. Note that

$$\text{SL}_{K_J}(V_J)^{\theta_{JL}} \leq \text{SL}_{K_L}(V_L)$$

if L is finite. One easily checks that $\theta_{JL}\theta_{LN} = \theta_{JN}$ if $\emptyset \neq J \subseteq L \subseteq N \subseteq I$.

For every non-empty, finite subset J of I , we put

$$G_J = \mathrm{SL}_{K_J}(V_J)^{\theta_{JI}} \leq \mathrm{GL}_K(V).$$

If J, L are non-empty, finite subsets of I with $J \subseteq L$, then

$$G_J = \mathrm{SL}_{K_J}(V_J)^{\theta_{JI}} = \mathrm{SL}_{K_J}(V_J)^{\theta_{JL}\theta_{LI}} \leq \mathrm{SL}_{K_L}(V_L)^{\theta_{LI}} = G_L$$

and $Z(G_J) \leq Z(G_L)$.

Put $G = \bigcup \{G_J \mid \emptyset \neq J \subseteq I, J \text{ finite}\}$. Then G is an infinite, locally finite group and $\{G_J \mid \emptyset \neq J \subseteq I, J \text{ finite}, \dim_{K_J} V_J \geq 3\}$ is a local system for G , contained in $\mathcal{P}$.

Let J be a non-empty, finite subset of I such that $\dim_{K_J} V_J \geq 3$. Pick a finite subset L of I containing J . Note that we view V_L as

$$(V_J \otimes_{K_J} K_L) \otimes_{K_L} (V_{L \setminus J} \otimes_{K_{L \setminus J}} K_L).$$

Hence $[V_L, G_J] \neq 0$. Let $\{f_1, \dots, f_n\}$ be a $K_{L \setminus J}$ -basis for $V_{L \setminus J}$. Then

$$V_L = \bigoplus_{i=1}^n (V_J \otimes_{K_J} K_L) \otimes_{K_L} (f_i \otimes_{K_{L \setminus J}} 1_{K_L}).$$

Note that $V_J \otimes_{K_J} K_L$ is an irreducible $K_L G_J$ -module. Hence every composition factor for G_J on V_L is isomorphic to $V_J \otimes_{K_J} K_L$ (and so non-trivial) and V_L is block-diagonal for G_J . Also G_J has $\dim_{K_{L \setminus J}} V_{L \setminus J}$ composition factors on V_L . So G_J has at least two non-trivial composition factors on V_L if $|L \setminus J| \geq 1$. By Theorem 2.1, $G/Z(G)$ is an LFS-group of p -type.

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Author information

Stefaan Delcroix, Department of Mathematics, California State University Fresno,
5245 N Backer Avenue M/S PB108, Fresno, Ca 93740, USA.

E-mail: sdelcroix@csufresno.edu