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Isotropic universe with almost scale-invariant fourth-order gravity

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We study a class of isotropic cosmologies in the fourth-order gravity with Lagrangians of the form $L = f(R) + k(G)$ where R and G are the Ricci and Gauss-Bonnet scalars, respectively. A general discussion is given on the conditions under which this gravitational Lagrangian is scale-invariant or *almost* scale-invariant. We then apply this general background to the specific case $L = \alpha R^2 + \beta G \ln G$ with constants α, β . We find closed form cosmological solutions for this case. One interesting feature of this choice of $f(R)$ and $k(G)$ is that for very small negative value of the parameter β , the Lagrangian $L = R^2/3 + \beta G \ln G$ leads to the replacement of the exact de Sitter solution coming from $L = R^2$ (which is a local attractor) to an exact, power-law inflation solution $a(t) = t^p = t^{-3/\beta}$ which is also a local attractor. This shows how one can modify the dynamics from de Sitter to power-law inflation by the addition of a $G \ln G$ -term. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4808255>]

I. INTRODUCTION

From the huge class of theories of gravitation which can be considered for describing and explaining the early evolution of the Universe, it is the subclass of scale-invariant ones which plays a prominent role. The reason for this prominence is that almost all physical theories and their resulting cosmologies have some limiting regime which is free from any scales.

In the present paper, we investigate a class of cosmologies which include not only scale-invariant theories but also *almost* scale-invariant theories. In Sec. II, we will present concrete definitions and results related to the several variants of “(almost) scale-invariant theories.”

The Lagrangians which we consider are of the form

$$L = f(R) + k(G), \quad (1.1)$$

where R is the curvature scalar and

$$G = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$$

is the Gauss-Bonnet scalar. By placing restrictions on the form of the functions $f(R)$ and $k(G)$, we will obtain theories which are scale-invariant and almost scale-invariant. We will focus on cosmological solutions to these scale-invariant and almost scale-invariant Lagrangians – in particular, spatially flat Friedmann space-times. We obtain general features for these almost scale-invariant cosmologies, and for certain cases we are able to completely integrate the resulting Friedmann-like equations to confirm older results and to obtain new exact, closed form solutions in terms of the Friedmann metric scale factor, $a(t)$.

Before moving to the detailed calculations, we give a brief review of work in this area which has some connection with the present paper. Cosmological models where the action depends on the

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Gauss-Bonnet scalar G are discussed in Refs. 1–4. In Ref. 5, exact solutions for $k(G) = G^\beta$ are given which have an ideal fluid source, a power law scale factor, $a(t) = t^p$, with p depending on β and the equation of state of the fluid; $a(t)$ is the related cosmic scale factor. Further papers on this topic are Refs. 6–14. In Ref. 15, the Λ CDM epoch reconstruction from $f(R, G)$ and modified Gauss-Bonnet gravities is presented. In this work, models with Lagrangians $R + k(G)$, or more general $f(R) + k(G)$, and also $R + \xi(\phi)G + \phi_i \phi^i$ are discussed especially for the spatially flat Friedmann models.

The paper¹⁶ investigates Λ CDM cosmological models, using Lagrangians of the form $L = k(G)$ and $L = R + k(G)$. For the pure $k(G)$ gravity, and a spatially flat Friedmann model with scale factor $a(t)$ where t is synchronized time, the following results are obtained: A de Sitter space-time with Hubble parameter $h = \frac{\dot{a}}{a} > 0$ has $G = 24h^4$. This de Sitter solution is a vacuum solution if the condition $Gdk/dG = k(G)$ is fulfilled. The exact power-law solution of the form $a(t) = t^p$ exists if the following condition is fulfilled:

$$0 = G \frac{dk}{dG} - k(G) + \frac{4G^2}{p-1} \cdot \frac{d^2k}{dG^2}, \quad (1.2)$$

i.e., if $k(G) = G^{(1-p)/4}$, or more completely the Euler-type equation (1.2) has solutions $k(G) = c_1 G + c_2 G^{(1-p)/4}$ with constants c_1 and c_2 – compare with Eqs. (59) and (60) of Ref. 16. The term $c_1 G$ is a divergence, and so does not contribute to the field equation, so, seemingly, only power-law Lagrangians $k(G) = G^{(1-p)/4}$ produce the exact solution $a(t) = t^p$. However, this is not the complete truth: If one takes the example $p = -3$, then the solutions of Eq. (1.2) become $k(G) = c_1 G + c_2 G \ln G$, a case not mentioned in Refs. 16. So, besides powers of G , also $G \ln G$ leads to exact solutions $a(t) = t^p$. Further recent papers on $k(G)$ gravity are Refs. 17–20. In Ref. 21, the Lagrangian

$$L = G \ln G \quad (1.3)$$

is discussed, and cosmological closed-form solutions are given, including the just mentioned exact solution $a(t) = t^p$ with $p = -3$.

In Ref. 22, the anomalous velocity curve of spiral galaxies is modelled by Lagrangians of type $L(R, G, \square G)$, where $\square$ denotes the D'Alembertian, especially in the form

$$L = \tilde{G} \ln \tilde{G}, \quad \text{where} \quad \tilde{G} = \frac{G}{(\square + \alpha R)G}. \quad (1.4)$$

In a first approximation, one can assume $\tilde{G} \approx G$, so this Lagrangian has similarity with that one from Eq. (1.3).

In Ref. 23, the stability of power-law solutions in cosmology is discussed for $L = G$, which gives non-trivial results for space-time dimension exceeding 4 only. In Ref. 24, solutions for $L = R + \sqrt{G}$ with a Friedmann scale factor of power-law form, i.e., $a(t) = t^p$ are given. In Ref. 25, the Lagrangian $L = R^n + \beta G^{n/2}$ is investigated. Further Lovelock models along the line of Ref. 25 are given in Ref. 26, whereas in Ref. 27 the case $k(G) = G^n + \beta G \ln G$ is discussed.

In Ref. 28, the stability of the cosmological solutions with matter in $f(R, G)$ gravity is discussed, with special emphasis on the stability of the de Sitter solution, and with Lagrangians of the type $R + R^n G^m$.

Analogous models for $f(R)$ -gravity can be found in Ref. 29, where the case $L = R^{3/2}$ is related to Mach's principle. In Ref. 30, an exact solution for $L = R^2$ is given. Further models are discussed in Refs. 31–36. In Ref. 37, the stability of models within theories of type $L = R + R^m + R^n$ with $n < 0 < m$ is discussed, and exact power-law solutions are obtained. Further papers on this topic are Refs. 38–42. In Ref. 43, the case $f(R) = R^{3/2}$ is studied. In Ref. 44, the following strict result is shown: Exact power-law cosmic expansion in $f(R)$ gravity models with perfect fluid as source is possible for $f(R) = R^n$ only. Newer models of this kind can be found in Refs. 45–81.

The conformal Weyl theory, especially the value of the perihelion advance in this theory, has been discussed in Refs. 82–91. For theories in lower-dimensional space-times, see, e.g., Refs. 92–94.

Our motivation for considering Lagrangians of the form given in Eq. (1.1) is as follows: We study the cosmological aspects of a specific version of $f(R, G)$ gravity which is scale-invariant in the sense that in the absence of matter no fundamental length exists within that theory. One can

contrast this with $R \pm l^2 R^2$ theories which have the fundamental length l . In connection with this we discuss and clarify that there are slightly different notions of scale-invariance, and we carefully distinguish between them. We do not insist on the second-order field equations,⁹⁵ so also nonlinear dependences of the Lagrangian on G are included with the result, that the field equations are of the fourth-order in general. Similarly, we do not motivate our research by the string theory,⁹⁶ but rather we want to present possible models for the observed evolution of the universe which includes both inflationary phase at early times and the present acceleration (normally attributed to some fluid/field generically termed “dark energy”) without the need to introduce additional matter fields. Thus our motivation is as follows: First, the leading principle is that in the first approximation, the Einstein-Hilbert Lagrangian R is the right one for weak fields. Second, a nonlinear addition to the Einstein-Hilbert Lagrangian depending on R , especially of the form R^2 or $R^2 \ln R$, gives the desired early time inflationary behaviour, see, e.g., the early papers³¹ on this topic. Third, the further addition of a term nonlinear in G to the Lagrangian was proposed in Refs. 37 and 3 and others as a possible alternative for dark energy, which is the generic term for the substance postulated to drive the current accelerated expansion of the Universe. To make the task tractable of finding which of these various modified gravity theories can give the observed late time acceleration the authors of Ref. 6 developed “the reconstruction program for the number of modified gravities including scalar-tensor theory, $f(R)$, $f(G)$, and string-inspired, scalar-Gauss-Bonnet gravity. The known (classical) universe expansion history is used for the explicit and successful reconstruction of some versions (of special form or with specific potentials) from all above modified gravities.”

The paper is organized as follows: As already said, in Sec. II we give a general discussion of scale-invariance and almost scale-invariance. This general discussion motivates our special choice

$$L = \alpha R^2 + \beta G \ln G \quad (1.5)$$

for the Lagrangian. Section III gives a brief, self-contained review of relevant formulas concerning the Gauss-Bonnet scalar and Gauss-Bonnet gravity. This is done since $k(G)$ models are much less known than $f(R)$ models. Section IV gives our main new results which follow from the almost scale-invariant Lagrangian of the form Eq. (1.5). Section V summarizes and gives conclusions about the results presented in this paper.

II. NOTIONS OF (ALMOST) SCALE-INVARIANCE

We first give the exact definitions of what we mean by an (almost) scale-invariant gravitational action or gravitational Lagrangian. A theory of gravitation with a geometric Lagrangian $L = L(g_{ij}, \partial_k)$ is defined by a scalar L which depends on the metric and its partial derivatives up to arbitrary order. The signature of the metric is $(- + \dots +)$ and $g = \det g_{ij}$. Within this section, we assume the dimension of space-time to be $D \geq 2$. Then the gravitational action I is defined by

$$I = \int L \sqrt{-g} d^D x. \quad (2.1)$$

A scale-transformation, also called a homothetic transformation, is a conformal transformation with a constant conformal factor. In another context, scale-transformations can also be interpreted as transformations that change the applied length unit. The transformed metric, $\tilde{g}_{ij}$, is defined as

$$\tilde{g}_{ij} = e^{2c} g_{ij}, \quad (2.2)$$

where c is an arbitrary constant. For the inverted metric, one gets

$$\tilde{g}^{ij} = e^{-2c} g^{ij}.$$

The Christoffel affinity Γ_{jk}^i , the Ricci tensor R_{ij} and the Riemann tensor R_{jkl}^i do not change under the scale-transformation given in Eq. (2.2), and also all covariant derivatives of the Ricci and Riemann tensors are homothetically invariant. However R , G , and g are changed under the transformation of Eq. (2.2.) as follows:

$$\tilde{R} = e^{-2c} R, \quad \tilde{G} = e^{-4c} G, \quad \tilde{g} = e^{2Dc} g. \quad (2.3)$$

Definition: The action (2.1) is called scale-invariant, if $\tilde{I} = I$ according to Eq. (2.2). It is called almost scale-invariant, if the difference $\tilde{I} - I$ is a topological invariant.

The Lagrangian L is called scale-invariant if there exists a constant m , such that

$$\tilde{L} \equiv L(\tilde{g}_{ij}) = e^{mc} L(g_{ij}). \quad (2.4)$$

Finally, the Lagrangian L is called almost scale-invariant, if the difference $\tilde{L} - e^{mc} L$ is a divergence.

Of course, the sum of a scale-invariant action and of an arbitrary topological invariant is always an almost scale-invariant action. Likewise, the sum of a scale-invariant Lagrangian and a divergence is always an almost scale-invariant Lagrangian. At first glance, one might be tempted to conclude that the converse should be true, that an almost scale-invariant action can always be written as the sum of a scale-invariant action plus a topological invariant and that an almost scale-invariant Lagrangian can always be written as the sum of a scale-invariant Lagrangian plus a divergence. However, as we will show below, there exist non-trivial examples of almost scale-invariant actions which cannot be represented in the form of such a sum.

The following relations between these four notions of scale-invariance exist: If L is scale-invariant, then with Eq. (2.3) we get

$$\tilde{I} \equiv \int \tilde{L} \sqrt{-\tilde{g}} d^D x = e^{(m+D)c} \int L \sqrt{-g} d^D x = e^{(m+D)c} I,$$

so for $m = -D$, a scale-invariant Lagrangian gives rise to a scale-invariant action.

Likewise for $m = -D$, an almost scale-invariant Lagrangian gives rise to an almost scale-invariant action, because the space-time integral of a divergence represents a topological invariant.

Let us now take the example $L = f(R)$ with an arbitrary but sufficiently smooth function f and ask, under which circumstances, this leads to scale-invariance. We have to distinguish two cases: $D = 2$ and $D > 2$. For $D = 2$, the scalar R represents a divergence, whereas for $D > 2$, no function of R has such a property.

Let us start with the more tractable case $D > 2$. As no function of R gives a divergence, the notions of scale-invariance and almost scale-invariance coincide. For $L = f(R)$ to be a scale-invariant Lagrangian there must exist an m such that the following relationship holds:

$$f(\tilde{R}) \equiv f(e^{-2c} R) = e^{mc} f(R) \quad (2.5)$$

using Eqs. (2.2)–(2.4). With f' denoting the derivative of f with respect to its argument we get from Eq. (2.5) by applying d/dc ,

$$-2f'(e^{-2c} R) \cdot e^{-2c} R = me^{mc} f(R).$$

Putting $c = 0$ into this equation we get a differential equation for $f(R)$,

$$-2Rf'(R) = mf(R), \quad (2.6)$$

which is solved by

$$f(R) = c_1 \cdot R^{-m/2} \quad (2.7)$$

with integration constant c_1 . As expected, just the powers of R lead to scale-invariant Lagrangians. The corresponding action I turns out to be scale-invariant for $m = -D$ only, i.e., $L = R^{D/2}$ leads to a scale-invariant action, for $D = 4$ this is the celebrated $L = R^2$.

Let us now turn to the less trivial case $D = 2$, where R represents a divergence. We look for the set of all almost scale-invariant Lagrangians. For a Lagrangian of the form $L = f(R)$ to be almost scale-invariant requires that there exists an m such that

$$f(\tilde{R}) \equiv f(e^{-2c} R) = e^{mc} f(R) + v(c) \cdot R, \quad (2.8)$$

where v depends on c only to ensure that $v \cdot R$ is a divergence for every c . Applying d/dc , we now get

$$-2f'(e^{-2c} R) \cdot e^{-2c} R = me^{mc} f(R) + v'(c) \cdot R.$$

Inserting $c = 0$ and abbreviating $v'(0)$ by c_2 we get in place of Eq. (2.6) now

$$-2Rf'(R) = mf(R) + c_2 \cdot R. \quad (2.9)$$

We divide by R , apply d/dR and get

$$-2\frac{d^2f}{dR^2} = \frac{d}{dR} \left(\frac{mf}{R} \right), \quad (2.10)$$

which is solved by

$$f(R) = c_3R + c_4R^{-m/2} \quad (2.11)$$

with integration constants c_3 and c_4 . This is just what one expected from the beginning: The divergence c_3R added to the power-law term $c_4R^{-m/2}$, i.e., the added divergence term $v(c) \cdot R$ in Eq. (2.8) leads to the extra divergence term c_3R in Eq. (2.11). However, Eq. (2.10) possesses a further solution besides Eq. (2.11): For $m = -2$ Eq. (2.10) is solved by

$$f(R) = c_3R + c_4R \ln R. \quad (2.12)$$

The result of Eq. (2.12) was already noted in Ref. 92: Besides what one would have expected, the action $I = \int R \ln R \sqrt{-g} d^2x$ turns out to be almost scale-invariant.

Now we perform the analogous analysis for the Lagrangian $L = k(G)$. For dimension $D \leq 3$, G vanishes, so this case is not interesting. For dimension $D \geq 5$, no function of G is a divergence, so we get the expected result: scale-invariance and almost scale-invariance coincide. Every power of G leads to a scale-invariant Lagrangian, and the action $I = \int G^{D/4} \sqrt{-g} d^Dx$ is scale-invariant.

So, $D = 4$ remains the only interesting case. Here, G represents a divergence, and we ask for the set of all almost scale-invariant Lagrangians. The condition that the Lagrangian $L = k(G)$ be almost scale-invariant means that there exists an m such that

$$k(\tilde{G}) \equiv k(e^{-4c}G) = e^{mc}k(G) + v(c) \cdot G. \quad (2.13)$$

Applying d/dc we now get

$$-4k'(e^{-2c}G) \cdot e^{-4c}G = me^{mc}k(G) + v'(c) \cdot G.$$

Inserting $c = 0$ and abbreviating $v'(0)$ by c_2 we get

$$-4Gk'(G) = mk(G) + c_2 \cdot G. \quad (2.14)$$

We divide by G , apply d/dG and get

$$-4\frac{d^2k}{dG^2} = \frac{d}{dG} \left(\frac{mk}{G} \right), \quad (2.15)$$

which is solved by

$$k(G) = c_3G + c_4G^{-m/4} \quad (2.16)$$

with constants c_3 and c_4 . This is just what one expects from the beginning: The divergence c_3G added to the power-law term $c_4G^{-m/4}$. However, Eq. (2.15) possesses one further solution besides Eq. (2.16): For $m = -4$, one gets

$$k(G) = c_3G + c_4G \ln G. \quad (2.17)$$

The result in Eq. (2.17), see Eq. (1.3), was already noted in Ref. 21: Besides, what one would have expected, the action $I = \int G \ln G \sqrt{-g} d^4x$ turns out to be almost scale-invariant.

An important property, valid not only for scale-invariant but also for almost scale-invariant Lagrangians is the following: If g_{ij} is a vacuum solution and $\tilde{g}_{ij}$ is homothetically related to g_{ij} , then $\tilde{g}_{ij}$ is also a vacuum solution. For the Lagrangians of type $L = f(R) + k(G)$ and dimension $D = 4$, only $L = \alpha R^2$ leads to a scale-invariant action, and only $L = \alpha R^2 + \beta G \ln G$ leads to an almost scale-invariant action. This is a strong argument for a further detailed study of the gravitational Lagrangian

$$L_g = \Lambda + R + \alpha R^2 + \beta G \ln G + \gamma C_{ijkl} C^{ijkl}. \quad (2.18)$$

Of course, the term $\gamma C_{ijkl} C^{ijkl}$ – see Ref. 90 – by itself has a scale-invariant action, but we did not consider it in this paper, as it has no influence on the field equation within the Friedmann models. The terms Λ and R are added here since in the weak field limit such terms appear effectively.

Einstein's theory of general relativity has a scale-invariant Lagrangian, but only if the cosmological term is absent (or is interpreted as part of the matter action), but not a scale-invariant action.

In closing this section, we note that one can construct scale-invariant Lagrangians which do not have the form $L = f(R) + k(G)$. One example is

$$L = R^{2n} \cdot f(G/R^2)$$

with a constant n and an arbitrary (transcendental) function f .

III. ON THE GAUSS-BONNET SCALAR

The field equations for the Lagrangian $L = f(R)$ are given by, see, for example, Eq. (2.27) of Ref. 42,

$$0 = L_R R_{ij} - g_{ij} L/2 + g_{ij} \square L_R - (L_R)_{;ij} \quad \text{where} \quad L_R = df/dR. \quad (3.1)$$

Since the case $L = f(R)$ has been widely studied, we will not go into further details here but simply refer the interested reader to the overview.⁴²

The case when $L = k(G)$ is much less known than the case $L = f(R)$ so we give some further details here. For the spatially flat Friedmann metric (given below in Eq. (4.1)), the Gauss-Bonnet scalar G becomes

$$G = 24h^2(h^2 + \dot{h}) = 24h^4(1 + \gamma), \quad (3.2)$$

where h is the Hubble parameter $h = \dot{a}/a$ and $\gamma = \dot{h}/h^2$. For the Lagrangian $k(G)$ with $k_G = dk/dG$, the corresponding vacuum field equation is given in Eq. (3.3) of Ref. 21 as

$$\begin{aligned} 0 = & \frac{1}{2} g^{ij} k(G) - 2k_G R R^{ij} + 4k_G R^i_k R^{kj} - 2k_G R^{iklm} R^j_{klm} \\ & - 4k_G R^{iklj} R_{kl} + 2R k_G^{;ij} - 2g^{ij} R \square k_G - 4R^{ik} k_{G;k}^{;j} \\ & - 4R^{jk} k_{G;k}^{;i} + 4R^{ij} \square k_G + 4g^{ij} R^{kl} k_{G;kl} - 4R^{ikjl} k_{G;kl}. \end{aligned} \quad (3.3)$$

See Eqs. (A4) and (A5) of Ref. 21, specialized to the space-time dimension $n = 4$,

$$\begin{aligned} R_{ijkl} = & C_{ijkl} + \frac{1}{4} (R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il}) \\ & - \frac{1}{6} R (g_{ik} g_{jl} - g_{il} g_{jk}). \end{aligned} \quad (3.4)$$

Here, C_{ijkl} is the Weyl tensor and we define $C^2 = C^{ijkl} C_{ijkl}$ and

$$Y^{ij} = R^{iklm} R^j_{klm} + 2R^{iklj} R_{kl}. \quad (3.5)$$

With this notation, the first (unnumbered) equation of the Appendix of Ref. 21 reads

$$C^{iklm} C_{jklm} = \frac{1}{4} \delta^i_j C^2. \quad (3.6)$$

Inserting Eqs. (3.4) and (3.6) into Eq. (3.5), we get

$$Y^{ij} = \frac{1}{4} g^{ij} C^2 + \frac{1}{6} R^2 g^{ij} - R R^{ij} - \frac{1}{2} g^{ij} R_{kl} R^{kl} + 2R^{ik} R^j_k. \quad (3.7)$$

Further it holds that

$$R_{ijkl} R^{ijkl} = C^2 + 2R_{kl} R^{kl} - \frac{1}{3} R^2 \quad (3.8)$$

and

$$G = C^2 - 2R_{kl}R^{kl} + \frac{2}{3}R^2. \quad (3.9)$$

With these notations we can rewrite Eq. (3.3) as

$$0 = \frac{1}{2}g^{ij}(k(G) - Gk_G) + 2Rk_G^{ij} - 2g^{ij}R\Box k_G - 4R^{ik}k_{G;k}^j - 4R^{jk}k_{G;k}^i + 4R^{ij}\Box k_G + 4g^{ij}R^{kl}k_{G;kl} - 4R^{ikjl}k_{G;kl}. \quad (3.10)$$

Inserting $k(G) = G^n$ into Eq. (3.10) we get with $k_G = nG^{n-1}$,

$$0 = -\frac{n-1}{2}g^{ij}G^n + 2nR(G^{n-1})^{ij} - 2ng^{ij}R\Box(G^{n-1}) - 4nR^{ik}(G^{n-1})_{;k}^j - 4n\left(R^{jk}(G^{n-1})_{;k}^i - R^{ij}\Box(G^{n-1}) - g^{ij}R^{kl}(G^{n-1})_{;kl} + R^{ikjl}(G^{n-1})_{;kl}\right). \quad (3.11)$$

Inserting $k(G) = G \cdot \ln G$ into Eq. (3.10), we get with $k_G = 1 + \ln G$,

$$0 = -\frac{1}{2}g^{ij}G + 2R(\ln G)^{ij} - 2g^{ij}R\Box(\ln G) - 4R^{ik}(\ln G)_{;k}^j - 4R^{jk}(\ln G)_{;k}^i + 4R^{ij}\Box(\ln G) + 4g^{ij}R^{kl}(\ln G)_{;kl} - 4R^{ikjl}(\ln G)_{;kl}. \quad (3.12)$$

IV. COSMOLOGICAL SOLUTIONS FOR $\alpha R^2 + \beta G \ln G$

In this section, we use the background developed in Secs. II and III to give a general study of spatially flat Friedmann space-times for almost scale-invariant Lagrangians. In particular, we focus on the case $L = \alpha R^2 + \beta G \ln G$ which the analysis of Sec. II pointed out as an important and unique case.

We start by setting up our system and notation. First, the cosmological metric we use is the spatially flat Friedmann space-time given as

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (4.1)$$

with positive cosmic scale factor $a(t)$. The dot denotes d/dt , $h = \dot{a}/a$ is the Hubble parameter, and $R = 6(2h^2 + \dot{h})$ is the curvature scalar.⁹⁷ Without loss of generality we assume $h \geq 0$. If this is not the case, then it is always possible to invert the time direction so as to get $h \geq 0$. If h appears in the denominator, this automatically includes the additional assumption, that $h \neq 0$.⁹⁸

It proves useful to define the function

$$\gamma = \dot{h}/h^2 = -\frac{d}{dt}\left(\frac{1}{h}\right), \quad (4.2)$$

which shall be used to replace $\dot{h}$ in subsequent formulas. In terms of γ we get $R = 6h^2(2 + \gamma)$. The deceleration parameter (i.e., $q = -\ddot{a}a/(\dot{a})^2$) is related to γ via $q = -1 - \gamma$.

Sometimes it proves useful to use $\tau = \ln a$ as an alternative time coordinate. With a dash denoting $d/d\tau$, we get with $\dot{\tau} = h$ the following formula:

$$\gamma' \equiv \frac{d\gamma}{d\tau} = \frac{d\gamma}{dt} \cdot \frac{dt}{d\tau} = \frac{\dot{\gamma}}{h}. \quad (4.3)$$

We now give some results which will be useful in dealing with the almost scale-invariant Lagrangians of the form given in Eq. (1.5). First we note that, assuming a spatially flat Friedmann metric of the form given in Eq. (4.1), that the vacuum field equation for a Lagrangian of the form $L = f(G, R)$, where F is a function of R and G , is (see Eq. (15) of Ref. 5),

$$0 = GF_G - F - 24h^3\dot{F}_G + 6(\dot{h} + h^2)F_R - 6h\dot{F}_R, \quad (4.4)$$

where $F_G = \partial F/\partial G$ and $F_R = \partial F/\partial R$. Equation (4.4) is the 00-component of the vacuum field equation for the Lagrangian $L = f(G, R)$. All other components of the vacuum field equation are

fulfilled if, Eq. (4.4) is valid. For the almost scale-invariant Lagrangian from Eq. (1.5) $L = \alpha R^2 + \beta G \ln G$ we get from Eq. (4.4),

$$0 = 3\alpha(1 + \gamma)(6\gamma + 3\gamma^2 + 2\gamma') - 2\beta(1 - 2\gamma - 3\gamma^2 - \gamma'). \quad (4.5)$$

We look for solutions of Eq. (4.5) with $\alpha\beta \neq 0$.

After some lengthy but straightforward calculations, it turned out that for the vacuum equation (4.4) following from Eq. (1.5) and restricting to the spatially flat Friedmann space-time Eq. (4.1), no cosmic bounce and no cosmic recollapse is possible; the proof was done by inserting a Taylor expansion for $a(t)$ into the field equation (4.4) and to show, that regular local extrema of this function do not exist. This result is not very surprising, as one knows this property to be valid already for both of the ingredients of Eq. (1.5), i.e., for $L = R^2$ and $L = G \ln G$. This fact simplifies the calculations as $h(t)$ cannot change its sign, and we do not need to match pieces of different sign of h together.⁹⁹

In a second step, we look for constant values $\gamma \neq 0$ related to the scale factor $a(t) = t^p$ representing the self-similar solutions.¹⁰⁰ To this end, we insert $\gamma = -1/p$ into Eq. (4.5). Without loss of generality we assume $\alpha = 1/3$ which transforms Eq. (4.5) to

$$0 = (1 - 1/p)(-6/p + 3/p^2) - 2\beta(1 + 2/p - 3/p^2). \quad (4.6)$$

This equation can be solved for β by

$$\beta = -3(2p - 1)/(2p(p + 3)). \quad (4.7)$$

As a first estimate we can see the following: The leading term in the limit $p \rightarrow \infty$ of Eq. (4.6) is $0 = -6/p - 2\beta$. If we insert $\beta = -3/p$ into Eq. (4.6), then we get qualitatively the following result: A very small negative value of the parameter β in the Lagrangian

$$L = \frac{1}{3} \cdot R^2 + \beta G \ln G \quad (4.8)$$

leads to the replacement of the exact de Sitter solution from $L = R^2$, which is a local attractor, to a PLI (power-law inflation) exact solution $a(t) = t^p = t^{-3/\beta}$ which also represents a local attractor. This shows how one can modify the dynamics from de Sitter to power-law inflation by the addition of the $G \ln G$ -term.

Let us now go more into the details of the system. The limit $\beta \rightarrow \infty$ in Eqs. (4.7)/(4.8) is essentially the limit to the Lagrangian $L = G \ln G$ from (4.8) and is related to the limit $p \rightarrow -3$ in Eq. (4.7), see the discussion near Eq. (1.3) for this case. At some points, the Cauchy problem fails to be well-posed: For $R = 0$, i.e., $\gamma = -2$, the field equation following from $L = R^2$ has the property that it is fulfilled by every space-time having $R \equiv 0$. Thus, the fourth-order field equation (3.1) possessing 10 independent components in the general case, now degenerates to one single second order scalar field equation, namely, $R = 0$.¹⁰¹ Further, for $G \leq 0$, the Lagrangian $G \ln G$ is not immediately defined; while this might be compensated by writing $G \ln |G|$ instead for $G < 0$, there remains to be a mild singularity at $G = 0$, i.e., for $\gamma = -1$. However, we are in the lucky circumstances that for those cosmological models where this model may play a role, i.e., near de Sitter and near to PLI behaviour, we have $G > 0$ anyhow. Therefore, we restrict the following discussion to the region $\gamma > -1$.

With $3\alpha = 1$, Eq. (4.5) can be rewritten as follows:

$$0 = 2\gamma'(1 + \beta + \gamma) + (1 + \gamma)(2\beta(3\gamma - 1) + 3\gamma(2 + \gamma)). \quad (4.9)$$

Here we meet the third case that the Cauchy problem is not well-posed: namely, at points where $1 + \beta + \gamma = 0$.¹⁰² We chose to restrict to that region where $1 + \beta + \gamma > 0$. For those cases, where $\gamma' = 0$, Eq. (4.9) can be solved for β by

$$\beta = -\frac{3\gamma(2 + \gamma)}{2(3\gamma - 1)}. \quad (4.10)$$

Analysing this Eq. (4.10), one can see that for a given value β , zero, one, or two related values γ exist. The range of values β , where no γ exists, is the interval

$$-2.2 \approx -(\sqrt{7} + 4)/3 < \beta < (\sqrt{7} - 4)/3 \approx -0.45.$$

In the third step, we see that Eq. (4.9) being of first order can be analysed qualitatively to see the asymptotic behaviour of the solutions: both for the past singularity as for the future expansion, the solutions tend to a solution with scale factor $a(t) = t^p$.

We count the degrees of freedom as follows: For a general solution of the fourth-order field equation and one free unknown, in this case $a(t)$, one expects 4 initial values, namely, $a(0)$ and the first three derivatives of $a(t)$ at $t = 0$. The remaining information is contained in the field equations. However, the 00-component of the field equation is a constraint, reducing the order by one. Thus the general solution is expected to have 3 free constants – one is a t -translation, the second is multiplication of t by some factor, the third is multiplication of $a(t)$ by a constant factor. In this sense, a general solution can be given even if it has no free parameter.¹⁰³

Let us sum the details: For every fixed $\beta > 0$, exactly three solutions exist. They can be described as follows: Find that value $\gamma(\beta)$ which has $0 < \gamma(\beta) < 1/3$ and solves Eq. (4.10). Then the solution with $\gamma = \gamma(\beta)$ is just the self-similar solution $a(t) = t^p$ with $p = -1/\gamma < -3$ discussed already earlier, see Eq. (4.7).¹⁰⁴ The second solution has $\gamma > \gamma(\beta)$ and $\gamma' < 0$ throughout, starts from an initial singularity with $\gamma \rightarrow \infty$ and attracts the self-similar solution $a(t) = t^p$ which, of course, reaches $a \rightarrow \infty$ in a finite time t . The third solution has $-1 < \gamma < \gamma(\beta)$ and $\gamma' > 0$ throughout, starts from an initial singularity of type $a(t) = t$ and also attracts the self-similar solution $a(t) = t^p$ for $a \rightarrow \infty$.

For $-0.45 \approx (\sqrt{7} - 4)/3 \leq \beta < 0$, we have essentially the same behaviour as in the previous case: One self-similar solution $a(t) = t^p$, but now with $p > 0$, and two other ones, both having $a(t) = t^p$ as attractor for $t \rightarrow \infty$. For the remaining negative values of β , the instabilities become more serious, and the behaviour of the solution has several different types of singularities. We conclude that probably the physically sensible range of our model is in the interval $\beta \geq (\sqrt{7} - 4)/3$.

In closing this section, we show that it is possible to completely integrate Eq. (4.5) for the special case when $\alpha = 0$; i.e., when $L = \beta G \ln G$ one can integrate the system completely to the point where one has an explicit form for the scale factor $a(t)$. This special case can be thought of as the limit where $\beta \gg \alpha$ so that system is dominated by the Gauss-Bonnet logarithmic term. In this case, taking $\alpha = 0$ Eq. (4.5) becomes $\gamma' = A\gamma^2 + B\gamma + C$ where the prime indicates differentiation with respect to τ and where A, B, C are

$$A = -3; B = -2; C = 1. \quad (4.11)$$

However, we will carry through the calculation until almost the end using general A, B, C since in this way the analysis can also be applied to the special Lagrangians $L = R^{2n}$ and $L = G^n$ which also lead to an equation for γ of the form $\gamma' = A\gamma^2 + B\gamma + C$. Integrating this equation for γ yields

$$\tau = \int \frac{d\gamma}{A\gamma^2 + B\gamma + C} = \frac{2}{D} \arctan\left(\frac{B + 2A\gamma}{D}\right), \quad (4.12)$$

where $D = \sqrt{-B^2 + 4AC}$. Inverting Eq. (4.11) gives

$$\gamma(\tau) = \frac{D}{2A} \tan\left(\frac{D\tau}{2}\right) - \frac{B}{2A}. \quad (4.13)$$

Now taking into account the form of A, B, C from Eq. (4.11) we find that $D = \sqrt{-B^2 + 4AC}$ is imaginary. Thus in Eq. (4.13), we replace D with iD_1 where $D_1 = \sqrt{B^2 - 4AC}$ and taking into account that $\tan(iD_1) = i \tanh(D_1)$ we find that Eq. (4.13) becomes

$$\gamma(\tau) = -\frac{D_1}{2A} \tanh\left(\frac{D_1\tau}{2}\right) - \frac{B}{2A}. \quad (4.14)$$

Next, we use this $\gamma(\tau)$ from Eq. (4.14) to solve the equation for $h = \dot{a}/a$ (where the overdot is differentiation with respect to t). First, we note that

$$\gamma(\tau) = -\frac{d}{dt} \left(\frac{1}{h} \right) = -\frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{1}{h} \right) = \frac{1}{h} \frac{dh}{d\tau}. \quad (4.15)$$

We have used $h = d\tau/dt$ in arriving at the final result. Now we integrate Eq. (4.15) for $\gamma(\tau)$ from Eq. (4.14) which yields

$$\ln(h(\tau)) = -\frac{1}{A} \ln \left[\cosh \left(\frac{D_1 \tau}{2} \right) \right] - \frac{B}{2A} \tau, \quad (4.16)$$

or solving for $h(\tau)$ gives

$$h(\tau) = \left[\cosh \left(\frac{D_1 \tau}{2} \right) \right]^{-1/A} \exp \left(-\frac{B}{2A} \tau \right). \quad (4.17)$$

Now using $h(t) = \dot{a}(t)/a(t)$ for the left hand side and $\tau = \ln[a(t)]$ in the right hand side of Eq. (4.17) gives

$$\begin{aligned} \frac{\dot{a}}{a} &= \left[\frac{1}{2} \left(\exp \left(\frac{D_1 \ln(a)}{2} \right) + \exp \left(-\frac{D_1 \ln(a)}{2} \right) \right) \right]^{-1/A} \exp \left(-\frac{B}{2A} \ln(a) \right) \\ &= \left(\frac{1}{2} \right)^{-1/A} (a^{(D_1+B)/2} + a^{(-D_1+B)/2})^{-1/A} \\ &= \left(\frac{1}{2} \right)^{1/3} (a + a^{-3})^{1/3}, \end{aligned} \quad (4.18)$$

where in the last line we have inserted the specific values of A, B, C, D_1 for this cases when $L = G \ln G$, i.e., $A = -3, B = -2, C = 1$, and $D_1 = 4$. Finally, integrating Eq. (4.18) gives

$$\int dt = t = 2^{1/3} \int a^{-1} (a + a^{-3})^{-1/3} da = 2^{1/3} a(t) {}_2F_1 \left(\frac{1}{4}, \frac{1}{3}; \frac{5}{4}; -a^4(t) \right) + k, \quad (4.19)$$

where k is an integration constant and ${}_2F_1(a, b; c; z)$ is the hypergeometric function. One should now solve Eq. (4.19) for $a(t)$ to obtain the scale factor as a function of t . Or a simpler method – given the presence of ${}_2F_1$ in Eq. (4.19) – is to plot t versus a and then flip the graph about the line $t = a$ thus graphically giving $a(t)$. If one does this, then one finds that $a(t)$ is an exponentially growing function of t , thus having a term like $G \ln G$ might be a way to obtain an early inflationary stage to the Universe.

V. DISCUSSION

In Sec. II, we had introduced the notions of scale-invariant and almost scale-invariant Lagrangians. For the set of Lagrangians $L = f(R) + k(G)$, we found out that L is scale-invariant if

$$L = \alpha R^{2n} + \beta G^n \quad (5.1)$$

with constants α, β , and n . This is a closed set of functions. If we add the divergence γG with constant γ , we arrive trivially at the following set of almost scale-invariant Lagrangians

$$L = \alpha R^{2n} + \beta G^n + \gamma G. \quad (5.2)$$

Of course, both Lagrangians give rise to the same set of field equations, but the 4-parameter set Eq. (5.2) fails to be a closed set of functions. To see this, we take the limit $n \rightarrow 1$ in Eq. (5.2), while α remains constant and $\beta = -\gamma = 1/(n-1)$. We get

$$\lim_{n \rightarrow 1} \alpha R^{2n} + G \cdot \frac{G^{n-1} - 1}{n-1} = \alpha R^2 + G \ln G.$$

In arriving at this result, we have defined $n-1 = \varepsilon$ and $G = e^z$ and used

$$\lim_{n \rightarrow 1} \frac{G^{n-1} - 1}{n-1} = \lim_{\varepsilon \rightarrow 0} \frac{G^\varepsilon - 1}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{e^{\varepsilon z} - 1}{\varepsilon} = z = \ln G.$$

Result: Besides the trivial almost scale-invariant Lagrangians, as defined by Eq. (5.2), the class of almost scale-invariant Lagrangians also includes the Lagrangians of the form¹⁰⁵

$$L = \alpha R^2 + \beta G \ln G + \gamma G. \quad (5.3)$$

Of course, the inclusion of matter will change the behaviour of the cosmological solutions discussed in this paper, but in the early stages of the Universe, with matter in the form of dust or radiation, the behaviour of the solutions above will only be marginally modified by the presence of the matter. The behaviour described in this paper will thus essentially correctly describe the dynamics.

Finally, let us reformulate one of the key results of this work given in Sec. IV which is closely related to analogous calculations done for higher dimensions in Refs. 23: For $p > 0$, the Lagrangian

$$L = \frac{1}{3} \cdot R^2 - \frac{3}{p+3} \cdot \frac{2p-1}{2p} \cdot G \ln G$$

has the spatially flat Friedmann with scale factor $a(t) = t^p$ as exact vacuum solution. For large values p , this is a local attractor solution and it represents a model for power-law inflation.

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- ⁹⁵ Of course, models with G entering linearly the Lagrangian became popular, because in dimensions larger than 4, they lead to second-order field equations, but this is not our goal here.
- ⁹⁶ Of course, others¹ mention just the connection with string theory as motivation.
- ⁹⁷ If the metric signature is changed to $(+ - - -)$, then R must be replaced by $-R$, whereas G remains the same. In regard to the ambiguities in the sign of R often found in the literature we note that we define R such that for the standard 2-sphere we always get $R > 0$.
- ⁹⁸ This is not a real restriction, as a constant function $a(t)$ is the trivial Minkowski space-time with $h \equiv 0$, and solutions, where $h(t) = 0$ at isolated points t only, can be matched by pieces with $h \neq 0$. In other words: if $h(t) = 0$ at isolated points t , then these are always connected to regions where $h(t) \neq 0$.
- ⁹⁹ For closed Friedmann models, however, such a behaviour is possible.
- ¹⁰⁰ A space-time is called self-similar, if scale-transformations can always be compensated by isometries.
- ¹⁰¹ Sometimes, that value of R where this happens, is called a critical value of the curvature scalar.
- ¹⁰² The only exception is the case $\beta = 0$, i.e., $L = R^2$, because for this case, Eq. (4.9) can be divided by $1 + \gamma$.
- ¹⁰³ It is possible to compensate both the multiplicative time and the multiplicative space as follows: The multiplication of t by a non-vanishing factor can be compensated by a scale-transformation of the metric bringing solutions to solutions as the field equation is scale-invariant, and the multiplication of a by a constant factor can be compensated by multiplication of x, y , and z by a constant factor. This latter possibility is a consequence of the fact that the Euclidean 3-space is self-similar. Neither the closed nor the open Friedmann models share this property.
- ¹⁰⁴ Of course, for $p < 0$, writing simply $a(t) = t^p$ may be misleading, because then $h < 0$ appears; but we believe that it is sufficiently clear that for $p < 0$, writing simply $a(t) = t^p$ means $a(t) = (t_0 - t)^p$ with $t < t_0$.
- ¹⁰⁵ We note that, since the argument of the logarithm in Eq. (5.3) should be positive and dimensionless we should replace $\ln G$ by $\ln(G/G_0)$ where G_0 is some constant $G_0 \neq 0$.