

## Various practice problems I

**Problem 1.**

Solve the following quadratic equations.

(a)  $8x^2 + 2x - 3 = 0$       (b)  $x^2 - 2x - 5 = 0$

**Problem 2.**

Solve the following inequalities.

(a)  $x^2 + 2x - 15 \geq 0$       (b)  $x^2 + x + 1 \leq 0$       (c)  $\frac{2x+1}{2-x} \leq 1$

**Problem 3.**Find the composite functions  $f \circ g$  and  $g \circ f$  where

$$f(x) = \sqrt{x+1} \quad \text{and} \quad g(x) = \frac{1}{x-1}.$$

**Simplify your answers as much as you can !****Problem 4.**If  $f(x) = 1/x$ , find and simplify

$$\frac{f(-1+h) - f(-1)}{h}, \quad \text{where } h \neq 0 \text{ and } h \neq 1.$$

**Problem 5.**

Sketch the graph of the following functions.

(a)  $f(x) = \sqrt{x+1} - 1$       (b)  $f(x) = |x-1| + 1$       (c)  $f(x) = \begin{cases} -2x+4 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2+2 & \text{if } x > 1 \end{cases}$       (d)  $f(x) = \begin{cases} |x-2| & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

**Problem 6.**Find an equation of the line that passes through the point  $(-1, 0)$  and is perpendicular to the line with the equation  $4x + 5y + 16 = 0$ .**Problem 7.**For  $f(x) = \frac{2x}{x+5}$  and  $g(x) = \frac{x}{3x-8}$ , find  $(f \circ g)(x)$  and  $(g \circ f)(x)$ . **Simplify your results !****Problem 8.**

Find the following limits.

(a)  $\lim_{x \rightarrow -1} \frac{3x^2 + 4x + 1}{x + 1}$       (b)  $\lim_{x \rightarrow -\infty} \frac{-2x^4 + 3x^3 - 7x - 10}{3x^4 + 6x^2 - x + 100}$

**Problem 9.**

Find the following limits

(a)  $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2}$       (b)  $\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$       (c)  $\lim_{x \rightarrow 1} \frac{\sqrt{x+3} - 2}{x - 1}$       (d)  $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3x^2}$

**Problem 10.**Determine the values of  $x$ , if any, at which the given function is discontinuous. At each point of discontinuity, state the condition(s) for continuity that are violated.

$$f(x) = \begin{cases} \frac{x^2-1}{x+1}, & \text{if } x \neq -1 \\ 1, & \text{if } x = -1. \end{cases}$$

**Problem 11.**Explain in details why  $f(x) = \begin{cases} 2x - 3, & \text{if } x \leq -1 \\ x^2 - 4, & \text{if } x > -1 \end{cases}$  is **NOT** continuous at  $x = -1$ .**Problem 12.**Let  $y = -x^2$ .(a) Find the average rate of change of  $y$  with respect to  $x$  on the interval  $[2, 3]$ .

(b) Find the (instantaneous) rate of change at  $x = 3$ .

**Problem 13.**

Find the derivatives of the following functions

$$(a) \quad g(s) = 2s^2 - \frac{4}{s} + \frac{2}{\sqrt{s}} \quad (b) \quad h(x) = \left(x + \frac{1}{x} + \frac{1}{x^2}\right)^5 \quad (c) \quad F(x) = \sqrt{\frac{x^2 + 1}{x^4 + 2}} + 10$$

**Problem 14.**

For the function

$$g(t) = \sqrt{2t^2 + 3}$$

find  $g''$  and  $g'''$ .

**Problem 15.**

Differentiate the following functions with respect to the indicated variable.

$$(a) \quad h(t) = \frac{t^2 - 3t + 1}{t + 1} \quad (b) \quad f(x) = \sqrt{c^2 x^2 + 2} \quad (c \text{ is a constant})$$

**Problem 16.**

Use the linear approximation to estimate  $(15)^{1/4}$

**Problem 17.**

Approximate  $f(x) = (1 + 2x)^{-n}$  at  $a = 0$  by the linear approximation. (Here,  $n$  is a positive integer.)

**Problem 18.**

The measurement of  $x$  is accurate within 2%. Use the linear approximation to determine the error  $\Delta f$  in the calculation of  $f$  and the percentage error  $100 \frac{\Delta f}{f}$  when

$$f(x) = \frac{1}{1 + x} \quad \text{and} \quad x = 4.$$

**Problem 19.**

A child is flying a kite. If the kite is 90 feet above child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the child is not moving and the cord remains straight from hand to kite, an unrealistic assumption.)

**Problem 20.**

A ladder 20 feet long leans against a building. If a bottom of the ladder slides away from the building horizontally at the rate of 2 ft/sec, how fast is the top of the ladder sliding down the building when the top of ladder is 12 feet above the ground.

**Problem 21.**

Use the linear approximation to estimate  $\sqrt[4]{81.6}$ .

**Problem 22.**

Assuming that the equator is a circle whose radius is approximately 4000 miles, how much longer than the equator would a concentric, coplanar circle be if each point on it were 2 feet above equator?

**Hint:** Use the linear approximation.

**Problem 23.**

Use the linear approximation to estimate  $f(x)$  at  $a$ .

$$f(x) = (1 - x)^{-n} \quad \text{at} \quad a = 0. \quad (\text{Assume that } n \text{ is a positive integer.})$$

**Problem 24.**

The speed  $v$  of blood flowing along the central axis of an artery of radius  $R$  is given by Poiseuille's law

$$v(R) = cR^2$$

where  $c$  is a constant. If you can determine the radius of the artery within an accuracy of 5%, how accurate is your calculation of the speed?

**Problem 25.**

(*Tilman's resource model*) Suppose that the rate of growth of a plant in a certain habitat depends on a single resource: for instance, nitrogen. Assume that the growth rate  $f(R)$  depends on the resource level  $R$  as

$$f(R) = a \frac{R}{k + R}$$

where  $a$  and  $k$  are constants. Express the percentage error of the growth rate,  $100\frac{\Delta f}{f}$ , as a function of the percentage error of resource level,  $100\frac{\Delta R}{R}$ .

**Problem 26.**

Determine (using derivatives) where  $f(x) = \frac{x}{x^2 + 1}$  is increasing and where is decreasing.

**Problem 27.**

In the following problems find the critical points and use the test you prefer to decide which give a local maximum value and which give a local minimum. What are these local maximum and local minimum values? Finally, find the inflection points.

$$(a) \quad f(x) = x^4 - 2x^2 + 3 \quad (b) \quad g(x) = x^4 + 2x^3 \quad (c) \quad h(x) = 2x + x^{\frac{2}{3}} \quad (d) \quad F(x) = \frac{x^2}{\sqrt{x^2 + 1}}$$

**Problem 28.**

Let  $f$  be a continuous function defined for all real numbers whose first derivative is

$$f'(x) = \frac{x-1}{3x^{2/3}} \text{ and whose second derivative is } f''(x) = \frac{x+2}{9x^{5/3}}.$$

- Find the critical points of  $f$ ;
- On what intervals is  $f$  increasing and on what intervals is  $f$  decreasing?
- Find local extrema of  $f$ ;
- On what intervals is  $f$  concave up and on what intervals is  $f$  concave down?
- Find points of inflection for  $f$ ;
- In addition, we know that  $f(0) = f(4) = 0$ . Use the information found in parts (a), (b), (c), (d), and (e) to sketch the graph of  $y = f(x)$ .

**Problem 29.**

For the function given by  $f(x) = (x^2 - 9)/(x^2 - 4)$  find

- The critical points of  $f$ ,
- The local maxima and the local minima of  $f$ ,
- The concavity of  $f$  and its points of inflections,
- All horizontal and vertical asymptotes,
- Finally, sketch the graph of the function.

**Problem 30.**

A man 6 feet tall is walking away from a street lights 18 feet high at a speed of 6ft/sec. How fast is the tip of his shadow moving along the ground?

**Problem 31.**

Determine where the graph of the given function is increasing, decreasing, concave up, and concave down. Find its local extrema. Then sketch its graph.

- $f(x) = x^6 - 3x^4$ .
- $g(x) = 3x^5 - 5x^3 + 1$ .
- $h(x) = x^{\frac{2}{3}}(1 - x)$ .

**Problem 32.**

If  $f'(x) = 2(x+2)(x+1)^2(x-2)^4(x-3)^3$ , what values of  $x$  make  $f(x)$  a local maximum? A local minimum?

**Problem 33.**

On the interval  $[0, 6]$ , sketch a *possible* graph of a continuous function that satisfies all of the stated conditions.

$$f(0) = 3; \quad f(3) = 0; \quad f(6) = 4;$$

$$f'(x) < 0 \text{ on } (0, 3); \quad f'(x) > 0 \text{ on } (3, 6);$$

$$f''(x) > 0 \text{ on } (0, 5); \quad f''(x) < 0 \text{ on } (5, 6).$$

**Note:** There are infinitely many graphs that satisfy all the conditions.

## Various practice problems II

**Problem 1.**

Suppose a rectangle has its lower base on the  $x$ -axis and upper vertices on the graph of the function  $y = 8 - x^2$ . Find the area of the largest such rectangle.

**Problem 2.**

A car rental agency has 24 identical cars. The owner of the agency finds that at a price of \$10 per day, all the cars are rented. However, for each \$1 increase in rental price, one of the cars is not rented. How much should be charged to maximize the income to the agency?

**Hint:** Let  $x$  denote the increase in rental above \$10. Find the income for car rental agency in terms of  $x$ .

**Problem 3.**

Sketch the graphs of the following functions. Label critical points, local extrema, points of inflection, and all asymptotes (vertical and horizontal).

(a)  $f(x) = \frac{8x}{x^2 + 4}$

(b)  $f(x) = x^{2/3}$

(c)  $f(x) = (x - 2)^{1/3}$

(d)  $f(x) = \frac{1}{x^5 + 1}$

**Problem 4.**

Decide whether the Mean Value Theorem applies to the given function on the given interval. If it does find all possible values of  $c$ ; if not, state the reason.

(a)  $f(x) = x^{5/3}; \quad [-1, 1]$

(b)  $g(x) = |x|; \quad [-2, 2]$

**Problem 5.**

Use the Mean Value Theorem to prove that

$$\lim_{x \rightarrow \infty} (\sqrt{x+2} - \sqrt{x}) = 0.$$

**Problem 6.**

Find the antiderivatives (i.e., indefinite integrals) of the given functions.

(a)  $\int \left(x - \frac{1}{x}\right)^2 dx$

(b)  $\int \frac{1}{\sqrt{3x+2}} dx$

(c)  $\int x(1-x^2)^{1/4} dx$

(d)  $\int \frac{(1 + \tan(x))^{1/3}}{\cos^2(x)} dx$

**Problem 7.**

Evaluate the following sums:

(a)  $\sum_{i=1}^5 [(3i+4)^{10} - (3i+1)^{10}],$

(b)  $\sum_{k=1}^n (3k^2 - 2k + 1).$

**Problem 8.**

Find the area under the curve  $y = x^3$  over the interval  $[0, 2]$ . To do this, divide the interval into  $n$  equal subintervals, calculate the area of the corresponding circumscribed polygon, and then let  $n \rightarrow \infty$ .

**Problem 9.**

Use **only** geometry to calculate  $\int_{-1}^2 (2 + 4x) dx$ .

**Problem 10.**

Evaluate  $\int_2^6 (x^2 + 6) dx$  using the definition of the definite integral. In other words, first evaluate the corresponding Riemann sum and then take a suitable limit.

**Problem 11.**

Evaluate

(a)  $\int_{-1}^3 (6x^2 - 1) dx$

(b)  $\int_2^3 x\sqrt{x^2 - 4} dx$

(c)  $\int_0^{\pi/6} \cos(2x) dx$ .

**Problem 12.**

In the following problems find  $G'(x)$

(a)  $G(x) = \int_5^x \cos(t^2) dt$       (b)  $G(x) = \int_1^{x^2+1} \sin(s)\sqrt{s} ds$       (c)  $G(x) = \int_{x^2}^3 x^3\sqrt{t^2 + 1} dt$

**Problem 13.**

Sketch the region bounded by the graphs of the given equations and calculate the area of the region.

(a)  $x = y^2 - 3y$ ,  $x - y + 3 = 0$

(b)  $y = x^2 - 4x + 3$ ,  $x - y - 1 = 0$

Sample Final Exam — Math. 150A, Fall 2010

**Problem 1.**

Find  $f'(x)$  for each of the following functions:

(a)  $f(x) = \frac{5x + 3}{\tan x}$

(b)  $f(x) = x \sin(4x^2 - 1)$

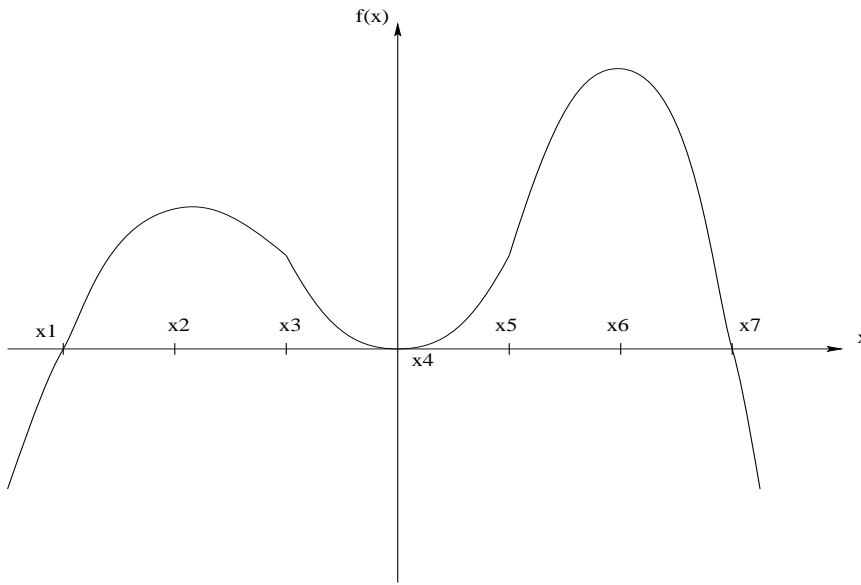
(d)  $f(x) = \int_0^{x^2+1} \frac{1}{t^{10} + 1} dt$

**Problem 2.**

Let  $f(x) = \begin{cases} kx + 5, & \text{if } x < 3; \\ c, & \text{if } x = 3; \\ kx^2, & \text{if } x > 3. \end{cases}$  (a) Choose the constant  $k$  so that  $\lim_{x \rightarrow 3} f(x)$  exists.  
 (b) Then choose the constant  $c$  so that  $f(x)$  is continuous at  $x = 3$ .

**Problem 3.**

For the function  $y = f(x)$  whose graph is given below what interval(s) correspond(s) to



(★)  $f'$  positive

$(-\infty, x_2) \cup (x_4, x_6) \iff$  Solved example.

(a)  $f'$  negative

(b)  $f''$  positive

(c)  $f''$  negative

(d)  $f'$  increasing

(e)  $f'$  decreasing

(f)  $f$  increasing

(g)  $f$  decreasing

**Problem 4.**

Find the equation of the tangent line to the graph of  $y = (x + 1) \sin x$  at  $x = 0$ .

**Problem 5.**

It is known that  $f$  is differentiable function such that  $f(2) = 9$  and  $f'(2) = 0.5$ . Use the method of tangent line approximation to estimate  $f(2.1)$ .

**Problem 6.**

Assuming that the equation  $x^2 + y^2 = x^3y$  gives  $y$  as a differentiable function of  $x$ , find  $\frac{dy}{dx}$ .

**Problem 7.**

The height above the ground of a certain object, thrown upward, is given by  $s(t) = -16t^2 + 128t + 32$ .

- Find its velocity at time  $t = 1$ .
- When is its velocity 0?
- Find the maximum height it reaches.
- When does it hit the ground?

**Problem 8.**

Find the area of the region bounded by the curve  $y = -x^2 - x - 2$  and the  $x$ -axis.

**Problem 9.**

Find the dimensions of the rectangle in the first quadrant with the largest area that has one side along the positive  $x$ -axis, one side along the positive  $y$ -axis, and one corner on the graph of  $y = 15 - 4x^2$ . Be sure to show that the point you find really gives a maximum rather than something else.

**Problem 10.**

Let  $f(x)$  be a continuous function defined for all real numbers whose first derivative is  $f'(x) = \frac{2-x}{x^{3/5}}$  and whose second derivative is  $f''(x) = -\frac{2}{5} \cdot \frac{(x+3)}{x^{8/5}}$ .

- Find the critical points of  $f$ .
- On what intervals is  $f$  increasing and on what intervals is  $f$  decreasing?
- Find local extrema of  $f$ .
- On what intervals is  $f$  concave up and on what intervals is  $f$  concave down?
- Find points of inflection of  $f$ .
- In addition, we know that  $f(0) = f(7) = 0$ . Use the information found in parts (a), (b), (c), (d), and (e) to sketch the graph of  $y = f(x)$ .

**Problem 11.**

Evaluate the following integrals.

- $\int_4^9 \frac{1 + \sqrt{x}}{\sqrt{x}} dx$
- $\int \frac{x+3}{x^2-9} dx$