PHYSICAL REVIEW E **87**, 060101(R) (2013)

Interacting fermions picture for dimer models

P. Falco
Department of Mathematics, California State University, Northridge, California 91330
(Received 25 December 2012; revised manuscript received 24 April 2013; published 6 June 2013)

Recent numerical results on classical dimers with weak aligning interactions have been theoretically justified via a Coulomb gas representation of the height random variable. Here, we propose a completely different representation, the interacting fermions picture, which avoids some difficulties of the Coulomb gas approach and provides a better account of the numerical findings. Besides, we observe that the Peierls argument explains the behavior of the system in the strong interaction case.

DOI: 10.1103/PhysRevE.87.060101 PACS number(s): 64.60.Cn, 05.50.+q, 71.27.+a

I. INTRODUCTION

The lattice model of hard-core close-packed dimers is among the most fundamental in two-dimensional statistical mechanics. Not only is it exactly solvable in the narrow sense (the free energy and many correlations can be computed [1–5]), but it also provides, through equivalences, the exact solutions of several other models, including the nearest neighbor Ising model [1,6], and some vertex models at the so called *free fermion point* [7,8].

Recently, especially in connection with a problem of quantum statistical mechanics [9], several authors have been studying the *classical* dimer model on a square lattice, modified by a weak aligning interaction. We will call it *interacting dimer model* (IDM). Since an exact solution of the IDM is not known, our knowledge of the properties of this system rests entirely on the numerical analysis of Refs. [10–13]; and on their theoretical interpretation via the *Coulomb gas approach* (CGA).

The CGA [14–16] (see also Ref. [17]) applies to every model for which one can define a *height random variable*; and it is based on the postulate that, in the scaling limit, height correlations are equal to charge correlations of the free boson field. This approach has been very successful in explaining long-range correlations in very many models of two-dimensional lattice statistical mechanics. From a practical viewpoint, though, the conjectured scaling limit of the height correlation has been difficult to substantiate (the best result in this direction is Ref. [18]). This implies that, for example, in the case of the IDM, the CGA has not provided so far: (a) an account of the staggered prefactors of dimer correlations; (b) the dependence of the critical exponents in the coupling constant of the model.

This Rapid Communication proposes an alternative method for studying the IDM, which we call *interacting fermions picture* (IFP). The basic idea is not new in physics. It has been a standard tool in condensed matter theory for studying the $1+1$-dimensional quantum models (see Refs. [19,20]). In classical two-dimensional statistical mechanics, it was first employed in Ref. [21] to demonstrate the universality of the nearest-neighbor Ising model under small, "solvability breaking" perturbations; subsequently, it was used to study the weak-universal properties of the eight-vertex and the Ashkin-Teller models [22,23]. In relation to the IDM, fermion viewpoints have already been employed in Refs. [24,25,27]; however, the IFP proposed here, as well as the results that we derive, appear to be completely new. The method is made of two steps: (i) the IDM is recasted into a lattice fermion field with a self-interaction; (ii) the scaling limit of the lattice field is shown to be the Thirring model, which is interacting as well, but which is also exactly solvable. The IFP solves the problems left open by the CGA because: (a) it clarifies the origin of the staggering prefactors of the dimers correlations; (b) it provides the relationship between the correlations critical exponents and the coupling constant as a series of Feynman graphs.

Before concluding, we also provide (by a different argument) a theoretical explanation of the numerical results in the strong interaction case.

II. DEFINITIONS AND RESULTS

Consider a finite box Λ of the infinite square lattice. A *dimer configuration*, ω, is a collection of dimers covering the edges of Λ with the constraint that every vertex of Λ is covered by one, and only one, dimer. The partition function of the IDM is

$$Z_\lambda(\Lambda) = \sum_\omega \exp\left[\lambda \sum_{d,d'\in\omega} v(d,d')\right], \tag{1}$$

where: λ is the dimers coupling constant; $v(d,d')$ is a two-body dimer interaction; the first sum is over all the dimer configurations; the second sum is over any pair of dimers in the configuration ω. In Ref. [11], $\lambda v(d,d')$ is a special, nearest-neighbor aligning interaction. In this work, we only assume that $v(d,d')$ is zero unless d and d' are both horizontal or both vertical, that it is invariant under $\pi/2$-rotations and under lattice translations, and that $|v(d,d')|$ has exponential decay in the distance between d and d'. The "noninteracting," exactly solvable dimer model is the case $\lambda = 0$ [1–5].

Our main result is the evaluation of correlation critical exponents of local bulk observables for small $|\lambda|$. A natural observable to consider is the *dimer occupancy* $\nu_d(\omega)$, which is equal to 1 if the dimer d is present in ω, and zero otherwise. Consider the horizontal dimers $d = \{0,\mathbf{e}_0\}$ and $d' = \{\mathbf{x},\mathbf{x}+\mathbf{e}_0\}$, for $\mathbf{e}_0 = (1,0)$ and $\mathbf{x} = (x_0,x_1)$. The IFP

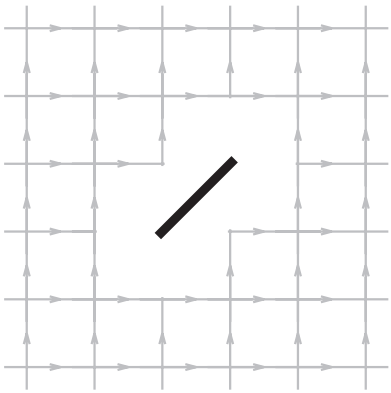

FIG. 1. Example of a diagonal dimer. Its presence in the graph generates a new face of clockwise oddness -1; thus, as opposed to the case of the monomer defect [3], the fermion representation of the diagonal dimer $\{\mathbf{x},\mathbf{x}+\mathbf{e}\}$ remains local and is $(-1)^{x_1+1} i\psi_{\mathbf{x}}\psi_{\mathbf{x}+\mathbf{e}}$.

provides the large-$|\mathbf{x}|$ formula

$$\langle \nu_d \nu_{d'}\rangle - \langle \nu_d\rangle\langle \nu_{d'}\rangle \sim (-1)^{x_0+x_1} c \frac{x_0^2-x_1^2}{\left(x_0^2+x_1^2\right)^2} + (-1)^{x_0} c_- \frac{1}{\left(x_0^2+x_1^2\right)^{\kappa_-}}, \qquad (2)$$

for a λ,v-dependent *critical exponent* $\kappa_- = 1 + O(\lambda)$ and *staggering prefactors* $(-1)^{x_0+x_1}$ and $(-1)^{x_0}$.

For $\lambda = 0$, this result coincides with the exact solution: see Eqs. (7.12) and (7.20) of Ref. [3]; and it agrees with the numerical simulations for small positive λ: see Eqs. (51) and (52) of Ref. [11].

The critical exponent κ_- is nonuniversal because it does depend on λ and $v(d,d')$. What is expected to be universal, instead, is the relationship among critical exponents of different observables. It is instructive to study a second observable, then. The authors of Ref. [11] considered the monomer correlation; however, being that it is equivalent to a *nonlocal* fermion correlation, the derivation of the scaling limit in the IFP is, at the present time, not more transparent than in the CGA. We consider, instead, a different observable, the *diagonal dimer*. It consists in a pair of monomers in positions $\{\mathbf{x},\mathbf{x}+\mathbf{e}\}$, where $\mathbf{e} = (1,1)$; see Fig. 1. Since such a dimer is not allowed in the hard-core, close-packed configurations ω, we define its "correlation" as it is done for the monomer observable, i.e., in terms of lattice defects:

$$\langle \nu_d\rangle = \lim_{\Lambda\to\infty} \frac{Z_\lambda(\Lambda - d)}{Z_\lambda(\Lambda)} \qquad \langle \nu_d \nu_{d'}\rangle = \lim_{\Lambda\to\infty} \frac{Z_\lambda[\Lambda - (d\cup d')]}{Z_\lambda(\Lambda)}.$$

For $d = \{0,\mathbf{e}\}$ and $d' = \{\mathbf{x},\mathbf{x}+\mathbf{e}\}$, the IFP gives the large-$|\mathbf{x}|$ formula

$$\langle \nu_d \nu_{d'}\rangle - \langle \nu_d\rangle\langle \nu_{d'}\rangle \sim c_+ \frac{(-1)^{x_0+x_1}-1}{\left(x_0^2+x_1^2\right)^{\kappa_+}}, \qquad (3)$$

for a new λ, v-dependent *critical exponent* $\kappa_+ = 1 + O(\lambda)$ and for a *staggering prefactor* $(-1)^{x_0+x_1} - 1$. The universal formula that relates κ_+ to κ_- is peculiar of the models with central charge $c = 1$ and was originally discovered (in a different model) by Kadanoff [26]:

$$\kappa_+\kappa_- = 1. \qquad (4)$$

In the following sections we will derive our main results: Eqs. (2), (3), and (4). As a byproduct, we will obtain a Feynman graphs representation of the expansion of $\kappa_\pm$ in powers of λ. For example, at first order [for $\hat v$ the Fourier transform of the interaction v—in Ref. [11], $\hat v(\mathbf{k}) = \cos k_0 + \cos k_1$],

$$\kappa_- = 1 + \frac{8}{\pi}\left[\hat v(\pi,\pi) - \hat v(0,\pi)\right]\lambda + O(\lambda^2); \qquad (5)$$

hence, according to the sign of $[\hat v(\pi,\pi) - \hat v(0,\pi)]\lambda$, either the former or the latter term in Eq. (2) is dominant at large distances. Note that the CGA was successfully used to justify the appearance of critical exponents $\kappa_\pm$, which satisfy Eq. (4) (see, for example, points 1 and 5 on page 7 of Ref. [27]). However, the CGA does not provide the staggering prefactors of Eqs. (2) and (3); nor does it provide any relationship between $\kappa_\pm$ and λ, such as Eq. (5) does.

III. INTERACTING FERMIONS PICTURE

When $\lambda = 0$, the dimer model is equivalent to a lattice fermion field without interaction. Namely,

$$Z_0(\Lambda) = \int D\psi \ \exp\left\{-\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}} K_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{x}}\psi_{\mathbf{y}}\right\}, \qquad (6)$$

where $\{\psi_{\mathbf{x}} : \mathbf{x}\in\Lambda\}$ are Grassmann variables and $D\psi$ indicates the integration with respect to all of them; $K_{\mathbf{x},\mathbf{y}}$ is the *Kasteleyn matrix*, which can be chosen to be such that

$$\sum_{\mathbf{y}} K_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{y}} = \sum_{\sigma=\pm1} \sigma\left(\psi_{\mathbf{x}+\sigma\mathbf{e}_0} + i\psi_{\mathbf{x}+\sigma\mathbf{e}_1}\right),$$

with $\mathbf{e}_0 = (1,0)$ and $\mathbf{e}_1 = (0,1)$. Equation (6) is the partition function of a free *Majorana fermion field*, i.e., a Grassmann-valued Gaussian field with moment generator

$$\langle e^{\sum_{\mathbf{x}}\psi_{\mathbf{x}}\eta_{\mathbf{x}}}\rangle_0 = e^{-\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}} S(\mathbf{x}-\mathbf{y})\eta_{\mathbf{x}}\eta_{\mathbf{y}}},$$

where the $\eta_{\mathbf{x}}$'s are other Grassmann variables, and S, the covariance, is the inverse Kasteleyn matrix

$$S(\mathbf{x}) = \langle\psi_{\mathbf{x}}\psi_0\rangle_0 = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dp_0}{2\pi}\int_{-\pi}^{\pi}\frac{dp_1}{2\pi}\ \frac{e^{ip_0x_0+ip_1x_1}}{i\sin p_0 - \sin p_1}.$$

The Fourier transform of S is singular at four *Fermi momenta*: $\mathbf{p}_{+,0} = (0,0)$, $\mathbf{p}_{+,1} = (\pi,\pi)$, $\mathbf{p}_{-,0} = (0,\pi)$, and $\mathbf{p}_{-,1} = (\pi,0)$. Therefore, in view of the scaling limit, it is convenient to decompose

$$\psi_{\mathbf{x}} = \sum_{\substack{\omega=\pm\\ s=0,1}} i^s e^{i\mathbf{p}_{\omega,s}\mathbf{x}}\psi_{\mathbf{x},\omega,s},$$

where $\psi_{\mathbf{x},\omega,s}$ are four *independent* Majorana fields with large-$|\mathbf{x}|$ covariances:

$$\langle\psi_{\mathbf{x},\omega,s}\psi_{0,\omega,s}\rangle_0 \sim \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\ \frac{e^{ip_0x_0+ip_1x_1}}{ip_0-\omega p_1} = \frac{1}{4\pi}\frac{1}{x_0+i\omega x_1}. \qquad (7)$$

This decomposition already appeared in Ref. [24] for studying the free case, which is exactly solvable. Instead, here we are preparing for the application to the interacting case and thus

we also introduce Dirac spinors $\psi^+_{\mathbf{x}} = (\psi^+_{\mathbf{x},+}, \psi^+_{\mathbf{x},-})$ and $\psi_{\mathbf{x}} = (\psi_{\mathbf{x},+}, \psi_{\mathbf{x},-})^T$ for

$$\psi^+_{\mathbf{x},\omega} = \frac{\psi_{\mathbf{x},\omega,0} + i\psi_{\mathbf{x},\omega,1}}{\sqrt{2}} \quad \psi_{\mathbf{x},\omega} = \frac{\psi_{\mathbf{x},\omega,0} - i\psi_{\mathbf{x},\omega,1}}{\sqrt{2}}, \tag{8}$$

with translational invariant covariances

$$\begin{aligned} &\langle \psi_{\mathbf{x},\omega}\psi_{0,\omega'}\rangle_0 = 0, \quad \langle \psi^+_{\mathbf{x},\omega}\psi^+_{0,\omega'}\rangle_0 = 0, \\ &\langle \psi^+_{\mathbf{x},\omega}\psi_{0,\omega'}\rangle_0 \sim \frac{\delta_{\omega,\omega'}}{4\pi}\frac{1}{x_0 + i\omega x_1}. \end{aligned} \tag{9}$$

If we now let $\lambda \neq 0$, by power series expansion in λ, one can verify that Eq. (1) becomes

$$Z_\lambda(\Lambda) = \int D\psi \ \exp\left[-\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}} K_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{x}}\psi_{\mathbf{y}} + V(\lambda v, \psi)\right], \tag{10}$$

where $V(\lambda v, \psi)$ is a sum of even monomials in the ψ's of order bigger than two. It is not difficult to see that the dimer correlation in the left-hand side of Eq. (2) becomes, in terms of Dirac fermions Eq. (8) and up to terms with faster decays

$$\begin{aligned} &\left\langle \psi_0\psi_{\mathbf{e}_0}\psi_{\mathbf{x}}\psi_{\mathbf{x}+\mathbf{e}_0}\right\rangle - \left\langle \psi_0\psi_{\mathbf{e}_0}\right\rangle\left\langle \psi_{\mathbf{x}}\psi_{\mathbf{x}+\mathbf{e}_0}\right\rangle \\ &\sim 4(-1)^{x_0+x_1}\sum_\omega \langle \psi^+_{0,\omega}\psi_{0,\omega}; \psi^+_{\mathbf{x},\omega}\psi_{\mathbf{x},\omega}\rangle^T \\ &\quad + 4(-1)^{x_0}\sum_\omega \langle \psi^+_{0,\omega}\psi_{0,-\omega}; \psi^+_{\mathbf{x},-\omega}\psi_{\mathbf{x},\omega}\rangle^T, \end{aligned} \tag{11}$$

where the label T indicates a truncated correlation. In the same way, the diagonal dimer correlation in the left-hand side of Eq. (3) becomes

$$\begin{aligned} &-(-1)^{x_1}\left[\langle \psi_0\psi_{\mathbf{e}}\psi_{\mathbf{x}}\psi_{\mathbf{x}+\mathbf{e}}\rangle - \langle \psi_0\psi_{\mathbf{e}}\rangle\langle \psi_{\mathbf{x}}\psi_{\mathbf{x}+\mathbf{e}}\rangle\right] \\ &\sim -8[(-1)^{x_0+x_1} - 1]\langle \psi^+_{0,+}\psi^+_{0,-}; \psi_{\mathbf{x},-}\psi_{\mathbf{x},+}\rangle^T. \end{aligned} \tag{12}$$

In the next section, by a renormalization group argument, we will explain why, in the evaluation of the large distance decay of the correlations, it is correct to replace the interacting fermion field Eq. (10) with the massless Thirring model. Assuming for the moment this crucial fact, we only need to borrow the exact solutions for the Thirring model correlations [28–30] (see also Ref. [31]):

$$\begin{aligned} \langle \psi^\dagger_\omega(0)\psi_\omega(0); \psi^\dagger_\omega(\mathbf{x})\psi_\omega(\mathbf{x})\rangle^T &= c\frac{x_0^2 - x_1^2}{\left(x_0^2 + x_1^2\right)^2} \\ \langle \psi^\dagger_+(0)\psi^\dagger_-(0); \psi_-(\mathbf{x})\psi_+(\mathbf{x})\rangle^T &= \frac{c_+}{\left(x_0^2 + x_1^2\right)^{\kappa_+}} \\ \langle \psi^\dagger_\omega(0)\psi_{-\omega}(0); \psi^\dagger_{-\omega}(\mathbf{x})\psi_\omega(\mathbf{x})\rangle^T &= \frac{c_-}{\left(x_0^2 + x_1^2\right)^{\kappa_-}}, \end{aligned} \tag{13}$$

where the critical exponents $\kappa_\pm$ are

$$\kappa_+ = \frac{1 + \frac{\lambda_T}{4\pi}}{1 - \frac{\lambda_T}{4\pi}} \quad \kappa_- = \frac{1 - \frac{\lambda_T}{4\pi}}{1 + \frac{\lambda_T}{4\pi}},$$

and λ_T is a parameter of the Thirring model: at first order, $\lambda_T = -16\left[\hat{v}(\pi,\pi) - \hat{v}(0,\pi)\right]\lambda + O(\lambda^2)$ (see next section). The derivation of Eqs. (2), (3), and (4) is complete.

IV. RG ANALYSIS

We follow Wilson's RG scheme in the version due to Gallavotti [32]. Integrating out the large momentum scales, one obtains an effective interaction

$$\begin{aligned} &\sum_n \sum_{\substack{\omega_1,\dots,\omega_n \\ s_1,\dots,s_{2n}}} \frac{i^{s_1+\cdots s_{2n}}}{(2\pi)^{4n-1}} \int d\mathbf{k}_1 \cdots d\mathbf{k}_{2n}\ \widehat{\psi}_{\mathbf{k}_1,\omega_1,s_1}\cdots\widehat{\psi}_{\mathbf{k}_{2n},\omega_{2n},s_{2n}} \\ &\cdot\delta\left(\sum_{j=1}^{2n}\mathbf{k}_j + \sum_{j=1}^{2n}\mathbf{p}_{\omega_j,s_j}\right)\widehat{w}_{2n}\left(\mathbf{k}_2 + \mathbf{p}_{\omega_2,s_2},\dots,\mathbf{k}_{2n} + \mathbf{p}_{\omega_{2n},s_{2n}}\right), \end{aligned}$$

where $\widehat{w}_{2n}$'s are a series of Feynman graphs. Some symmetries are of crucial importance. For $R(k_0,k_1) = (k_1, -k_0)$, $\vartheta(k_0,k_1) = (k_1,k_0)$, and $\tau(k_0,k_1) = (k_0,k_1+\pi)$, we find

$$\begin{aligned} \widehat{w}_{2m}(\tau\mathbf{k}_2,\dots,\tau\mathbf{k}_{2m}) &= (-i)^m\widehat{w}_{2m}(\vartheta\mathbf{k}_2,\dots,\vartheta\mathbf{k}_{2m}) \\ \widehat{w}_{2m}(R\mathbf{k}_2,\dots,R\mathbf{k}_{2m}) &= (-i)^m\widehat{w}_{2m}(\mathbf{k}_2,\dots,\mathbf{k}_{2m}) \\ \widehat{w}_{2m}(\mathbf{k}_2,\dots,\mathbf{k}_{2m})^* &= i^m\widehat{w}_{2m}(\vartheta\mathbf{k}_2,\dots,\vartheta\mathbf{k}_{2m}). \end{aligned} \tag{14}$$

From power counting, there are two possible local, marginal terms: a quartic term, that requires the renormalization of the coupling constant λ; a quadratic term, responsible for a field renormalization. By Eq. (14) they are

$$24\widehat{w}_4(\mathbf{p}_{+,1},\mathbf{p}_{-,0},\mathbf{p}_{-,1})\sum_{\mathbf{x}}\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-} \tag{15}$$

and

$$2\left[-i\frac{\partial\widehat{w}_2}{\partial k_0}(\mathbf{p}_{+,0}) - \frac{\partial\widehat{w}_2}{\partial k_1}(\mathbf{p}_{+,0})\right]\sum_{\mathbf{x},\omega}\psi^+_{\mathbf{x},\omega}\partial_\omega\psi^-_{\mathbf{x},\omega}, \tag{16}$$

where ∂_ω is the Fourier transform of $ik_0 - \omega k_1$. Again by Eq. (14), the prefactors in Eqs. (15) and (16) are real. Instead, there are no local, relevant terms: the only possible one, a quadratic term without derivatives, i.e., a *mass term*, cannot be generated by conservation of total momentum. These facts imply that Eq. (10), with parameter λ, equals the massless Thirring model, with parameter λ_T, up to terms which are irrelevant and thus cannot modify Eq. (4) (although they do contribute to the relationship between λ and λ_T as series of Feynman graphs).

V. STRONG INTERACTION

In the opposite case of *strong* dimer interaction, i.e., $|\lambda| \gg 1$, the numerical findings of Ref. [11] indicate the existence of four different "columnar" phases. Also, this outcome can be theoretically explained, not by the IFP this time but rather by the classical Peierls argument. However, as opposed to the weak interaction case, the outcome does depend

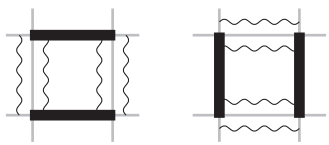

FIG. 2. Draw four wiggly lines for each plaquette containing two facing dimers. An equivalent but completely local way of evaluating the total energy of a dimer configuration is then by assigning an energy of $-\frac{\lambda}{4}$ per wiggly line.

on the choice of $v(d,d')$: for definiteness, we discuss here the choice in Ref. [11], which assigns an energy $-\lambda < 0$ per each plaquette displaying one of the two dimer arrangements in Fig. 2. Decorate a dimer configuration with wiggly lines as indicated in Fig. 2: a plaquette is "good" if it contains two wiggly lines; otherwise, the plaquette is "bad." Note that: (a) dimer configurations on nearest-neighbor good plaquettes must correspond to the same columnar ground state; (b) the probability of a bad plaquettes occurrence is dumped, at least, by a factor $e^{-\frac{\lambda}{4}}$ per plaquette. Therefore, one can apply the Peierls argument to the "contours" of bad plaquettes (see Refs. [33,34] or the review in Ref. [35]) and show that, for λ positive and large, the number of different phases coincides with the number of ground states.

VI. CONCLUSION

In the case of weak, short-ranged interactions, we have shown that the IFP provides a detailed account of the numerical findings of Ref. [11]—plus some new predictions. This method should also work, with possibly different outcomes, for triangular and hexagonal lattices; and for two or more interacting copies of dimer models. Besides, the IFP should be applicable to the six-vertex model, which is equivalent to dimers on a square lattice with a staggered interaction. Including the results on Ashkin-Teller, eight-vertex, and XYZ quantum chain [23], the IFP seems quite an effective way for dealing with two-dimensional lattice critical models with central charge $c = 1$. It might be possible that the IFP be applicable to $c < 1$ models (for example via ideas in Ref. [15]).

In the case of strong aligning interactions, we have explained the numerical findings of Ref. [11] by the Peierls argument.

ACKNOWLEDGMENTS

The author thanks T. Spencer, M. Biskup, and F. Bonetto for helpful discussions.

[1] P. Kasteleyn, J. Math. Phys. **4**, 287 (1963).
[2] E. Lieb, J. Math. Phys. **8**, 2339 (1967).
[3] M. Fisher and J. Stephenson, Phys. Rev. **132**, 1411 (1963).
[4] R. Kenyon, Ann. Inst. Henri Poincaré Probab. Stat **33**, 591 (1997).
[5] J. Dubédat, arXiv:1110.2808 (2011).
[6] M. Fisher, J. Math. Phys. **7**, 1776 (1966).
[7] C. Fan and F. Wu, Phys. Rev. B **2**, 723 (1970).
[8] F. Wu and K. Lin, Phys. Rev. B **12**, 419 (1975).
[9] D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[10] F. Alet, J. L. Jacobsen, G. Misguich, V. Pasquier, F. Mila, and M. Troyer, Phys. Rev. Lett. **94**, 235702 (2005).
[11] F. Alet, Y. Ikhlef, J. L. Jacobsen, G. Misguich, and V. Pasquier, Phys. Rev. E **74**, 041124 (2006).
[12] F. Trousselet, P. Pujol, F. Alet, and D. Poilblanc, Phys. Rev. E **76**, 041125 (2007).
[13] H. Otsuka, Phys. Rev. E **80**, 011140 (2009).
[14] L. Kadanoff, J. Phys. A **11**, 1399 (1978).
[15] M. den Nijs, Phys. Rev. B **27**, 1674 (1983).
[16] B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).
[17] J. Jacobsen, in *Polygons, Polyominoes and Polycubes*, Lect. Notes Phys., edited by A. J. Guttmann, Vol. 775 (Springer, Berlin, 2009), pp. 347–424.
[18] J. Fröhlich and T. Spencer, Phys. Rev. Lett. **46**, 1006 (1981).
[19] J. Solyom, Adv. Phys. **28**, 201 (1979).
[20] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, New York, 2004).
[21] T. Spencer, J. Phys. A **279**, 250 (2000); H. Pinson and T. Spencer (unpublished) (2000).
[22] A. Giuliani and V. Mastropietro, Phys. Rev. Lett. **93**, 190603 (2004).
[23] G. Benfatto, P. Falco, and V. Mastropietro, Phys. Rev. Lett. **104**, 075701 (2010).
[24] P. Fendley, R. Moessner, and S. L. Sondhi, Phys. Rev. B **66**, 214513 (2002).
[25] H. Yao and S. A. Kivelson, Phys. Rev. Lett. **108**, 247206 (2012).
[26] L. P. Kadanoff, Phys. Rev. Lett. **39**, 903 (1977).
[27] S. Papanikolaou, E. Luijten, and E. Fradkin, Phys. Rev. B **76**, 134514 (2007).
[28] K. Johnson, N. Cimento **20**, 773 (1961).
[29] B. Klaiber, Helv. Phys. Acta **37**, 554 (1964).
[30] C. Hagen, N. Cimento B **51**, 169 (1967).
[31] P. Falco, arXiv:1208.6568 (2012).
[32] G. Gallavotti, Rev. Mod. Phys. **57**, 471 (1985).
[33] O. J. Heilmann and E. H. Lieb, J. Stat. Phys. **20**, 679 (1979).
[34] J. Fröhlich, R. Israel, E. Lieb, and B. Simon, J. Stat. Phys. **22**, 297 (1980).
[35] M. Biskup, in *Methods of Contemporary Mathematical Statistical Physics*, Lect. Notes Math., edited by R. Kotecky, Vol. 1970 (Springer, Berlin, 2009).