Minkowski’s Theorem for Incomplete Tropical Fans

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by
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Certification of Approval

I certify that I have read Minkowski’s Theorem for Incomplete Tropical Fans by Ian Xander Wallace and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Arts at San Francisco State University.

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Abstract

A classical theorem of Minkowski establishes a balancing condition on the unit outer normal vectors to the faces of a polytope. A reinterpretation of this theorem to fans says that face volumes of a given polytope give Minkowski weights on its normal fan. This theorem only applies to complete fans, so in this thesis, we lay the groundwork for a generalization of this theorem to incomplete tropical fans. For a given incomplete fan, we obtain a family of polytopal complexes called normal complexes that have the same relationship with their associated fan that polytopes have with their normal fans. We restrict to the 2-dimensional case, and we show that the 1-dimensional face volume of a normal complex is a Minkowski 1-weight on its associated fan.
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Table of Contents

Table of Contents vi

List of Figures vii

1 Shapes 4
  1.1 Polytopes and Minkowski’s Classical Theorem 4
  1.2 Fans 14
  1.3 Normal Complexes 24
  1.4 Volumes of Normal Complexes 29
  1.5 Faces and Face Volumes 32

2 Minkowski’s Theorem for 2-D Tropical Fans 35
  2.1 Facets of 2-Dimensional Normal Complexes 36
  2.2 Minkowski Weights for Incomplete 2-Fans 44
  2.3 Related and Future Work 47

Bibliography 51
## List of Figures

0.1 An example of a normal complex that we will run into more than once. ........... 2

1.1 Depictions of unit outer normal vectors of faces of a cube, translated to their respective faces. ......................................................... 9

1.2 A 2-dimensional example, showing the unit simplex in $H_1$ being compressed under projection onto $H_2$. ....................................................... 11

1.3 Left: $u_1 \ast \widetilde{u}_1 = v \ast u$. Right: $u_1 \ast \widetilde{u}_1 = -(v \ast u)$. ....................... 12

1.4 A 3-dimensional cone $\sigma$ with a highlighted faces $\tau$ and $\sigma \setminus \tau$. ....................... 18

1.5 The simplex $P$ with unit outer normal vectors translated to their respective facets. 20

1.6 We will refer back to this balanced fan a few times to build further concepts. ...... 24

1.7 Notice that $H_{\rho_1,*}(z)$ and $H_{\rho_2,*}(z)$ intersect within $\sigma$. ....................... 25

1.8 Here, $u_{\rho_1} = (1, 3)$, $u_{\rho_2} = (3, 1)$, and $z = (6, 2)$. ....................... 27

1.9 This normal complex is informally called a “fletching,” which is the feathery back-end of a dart/arrow. The name was coined by Lauren Nowak. ............... 28

1.10 The quadrilateral $P_{\sigma,*}(z)$, with the unit-volume simplex overlaid. ............... 30

1.11 The red facet is the intersection of the normal complex $C_{\sigma,*}(z)$ with one of its supporting hyperplanes $H_{\rho,*}(z)$. ....................... 33
Introduction

A classical theorem of Hermann Minkowski gives a way to establish a balancing condition on the unit outer normal vectors of the faces of a polytope by weighting them with the volume of their respective faces [8]. This thesis is centered around this theorem and aims to generalize it to a broader class of polytopal structures called normal complexes.

Here is our synopsis: we like to weight things so they balance. This raises some questions. In what sense are we weighting things? What things are we weighting? What do we mean by balancing? At first, in light of Minkowski’s classical theorem, the things we weight are normal vectors to faces of a given polytope, the weights are the volumes of said faces, and by balancing, we mean that the weighted sum of those normal vectors is the zero vector. After recognizing the relationship between polytopes and polyhedral fans via normal fans of polytopes, we reinterpret the classical theorem by weighting cones of a fan by the volumes of faces of a given associated polytope. In this case, we want a balancing condition on the ray generators of the fan associated to the given polytope. This balancing condition should capture the same information as in the statement of Minkowski’s classical theorem, but only in terms of its normal fan. Normal fans of polytopes are complete, so this theorem as it stands really only gives us information about complete fans. This thesis is about removing that restriction.

We want a weighted sum of ray generators of an incomplete fan to satisfy a balancing condition as in Minkowski’s theorem, and the weights should be volumes of faces of something
associated to that fan, but if we’re starting with an incomplete fan, we don’t have a polytope to compute volumes of. We need to invoke some polytopal structure whose normal fan is incomplete. This is where we meet normal complexes.

Figure 0.1: An example of a normal complex that we will run into more than once.

Normal complexes are polytopal complexes first introduced by Nathanson and Ross in 2021 [6]. These work great for our cause, since they have incomplete normal fans, and we have methods for computing their volumes and the volumes of their faces, as given by Nowak, O’Melveny, and Ross in 2022 [7]. In particular, we are particularly interested in normal complexes associated to tropical fans, which already have a pre-established balancing condition on them.

The first half of this thesis is about familiarizing ourselves with the related notation and objects motivated toward Minkowski’s theorem. We prove Minkowski’s theorem in terms of polytopes and their face volume, then rephrase it in terms of normal fans of polytopes. Once we know how Minkowski’s theorem works for complete fans, we aim to extend the
theorem by defining normal complexes associated to incomplete tropical fans and getting a notion of volume for them. The second half of this thesis pertains to the groundwork toward removing the restriction of the theorem applying to complete fans. Since these are first steps toward a general conjecture about incomplete tropical fans, we restrict to the 2-dimensional case, which is the lowest-dimensional, nontrivial case of this conjecture. For simplicity, we also restrict to simplicial fans, since the aforementioned normal complexes require them for their definition. The content of this portion of the thesis includes constructing a closed formula for facet volumes of 2-dimensional normal complexes (2.1), a proof showing that the volume formula satisfies our balancing condition (2.1), and a brief discussion about related and ongoing work.
Chapter 1

Shapes

The central objects of this project are normal complexes, and the central theorem of this project relates to the volumes of the faces of normal complexes. Normal complexes and their volumes come from polyhedra and their volumes. In this chapter, we build these concepts.

1.1 Polytopes and Minkowski’s Classical Theorem

Polyhedra can be naively thought of as ‘shapes with flat sides.’ To make sense of this idea, we first need a notion of ‘flatness.’ Let $V$ be a real, finite-dimensional inner product space with inner product $u \ast v$ for $u, v \in V$.

Definition 1.1. An affine subspace of $V$ is a subset of $V$ of the form $v + U$, where $v \in V$ and $U$ is a linear subspace of $V$. An affine hyperplane $H$ is an affine subspace of $V$ given
by
\[ H := \{ v \in V : a \ast v = b \} \]
for some \( a \in V \setminus \{\vec{0}\} \) and some \( b \in \mathbb{R} \). \( H \) is a \textbf{linear} hyperplane if \( b = 0 \). We define the \textbf{closed halfspaces} associated to an affine hyperplane \( H \) as
\[ H^+ := \{ v \in V : a \ast v \geq b \} \quad \text{and} \quad H^- := \{ v \in V : a \ast v \leq b \}. \]
Since any inequality can be reversed via multiplication by \(-1\), we will use \( H^- \) to denote a closed halfspace.

A hyperplane \( H \) is an affine (or linear) subspace of \( V \) of codimension 1. The hyperplane \( H \) cuts \( V \) cleanly in half, those halves being given by \( H^+ \) and \( H^- \). Hyperplanes are the notion of ‘flatness’ that we need to define polyhedra and polytopes.

\textbf{Definition 1.2.} A \textbf{polyhedron} \( P \subseteq V \) is the finite intersection of closed halfspaces. A \textbf{polytope} is a bounded polyhedron.

\textbf{Example 1.1.} To illustrate this definition, let \( \ast \) denote the dot product on \( \mathbb{R}^2 \) and consider the following hyperplanes in \( \mathbb{R}^2 \):
\[ H_1 = \{(x, y) \in \mathbb{R}^2 : (x, y) \ast (-1, 0) = 1\}, \]
\[ H_2 = \{(x, y) \in \mathbb{R}^2 : (x, y) \ast (0, -1) = 1\}, \quad \text{and} \]
\[ H_3 = \{(x, y) \in \mathbb{R}^2 : (x, y) \ast (1, 1) = 1\}. \]
Now consider their associated closed halfspaces $H_1^-$, $H_2^-$, and $H_3^-$:

The intersection $H_1^- \cap H_2^- \cap H_3^-$ gives us a triangle with vertices at $(-1, -1)$, $(-1, 2)$, and $(2, -1)$.
A key concept in this paper and in every subfield of geometry is the idea of the dimension of a set, which we have invoked previously, but should be given a concrete definition. For the following, note that for any affine subspace \( A \subseteq V \), there exists a unique linear subspace \( U \subseteq V \) such that \( U + v = A \) for some \( v \in V \). For brevity, given an affine subspace \( A \subseteq V \), denote its unique associated linear subspace by \( A' \).

**Definition 1.3.** The **dimension of an affine space** \( A \subseteq V \), denoted \( \dim(A) \), is the dimension of its associated linear subspace \( A' \). By convention, if \( A = \emptyset \), then \( \dim(A) = -1 \).

So this gives us a way to tell how big an affine subspace is, but how do we relate this to polyhedra? Any subset of \( V \) has some affine subspace containing it, such as \( V \) itself. However, some sets in \( V \) may be contained in affine subspaces of lower dimension than that of \( V \). To define the dimension of a set, we should aim to minimize the dimension of the affine space containing it.

**Definition 1.4.** Given a set \( P \subseteq V \), define its **affine hull** as

\[
\text{aff}(P) := \bigcap \{ \text{affine spaces } H \subseteq V \text{ such that } P \subseteq H \}.
\]

That is, the affine hull of \( P \) is the smallest affine space containing \( P \). The **dimension of a polyhedron** \( P \) is defined as \( \dim(P) := \dim(\text{aff}(P)) \).

For instance, the triangle as in the prior example (1.1) is a subset of \( \mathbb{R}^2 \), but not of any hyperplane of \( \mathbb{R}^2 \), hence its dimension is 2.
CHAPTER 1. SHAPES

Recall that we started by naively characterizing polyhedra as ‘shapes with flat sides.’ To make that idea rigorous, we first specify the halfspaces that support a polyhedron.

**Definition 1.5.** A hyperplane $H$ supports $P \subseteq V$ if $P \subset H^-$. We call $H$ a supporting hyperplane, and $H^-$ a supporting halfspace.

In example (1.1), the given hyperplanes are special examples of supporting hyperplanes that contain the ‘sides’ of the resulting triangle, which we have been alluding to. Here, we can properly define them.

**Definition 1.6.** A face of a polyhedron $P$ is a subset $F \subseteq P$ given by $P \cap H$, where $H$ is a supporting hyperplane of $H$. A facet of $P$ is a codimension 1 face of $P$, and a vertex of $P$ is a 0-dimensional face of $P$. Note that the affine hull of any facet is a supporting hyperplane.

For any polyhedron $P$, denote by $\hat{P}$ the set of all faces of $P$.

We now know enough about polytopes to invoke the motivation behind this paper, but we’re missing two important pieces of the puzzle, one being the unit outer normal vector to a given face. A normal vector to a hyperplane $H$ is a vector $v$ such that for all $u \in H$, $u \ast v = 0$. A unit (or unit-length) vector is a vector $v$ such that $v \ast v = 1$. Normal vectors only exist for hyperplanes (or codimension 1 surfaces), so for lower dimension affine sets, we restrict to their affine subspaces. Given a supporting hyperplane of a polytope, there exist two normal vectors: one that ‘points in’ to the polytope, and one that ‘points out.’ The latter is the one we use for Minkowski’s theorem, so it calls for a proper definition. Let
[u, v] denote the closed line segment between two vectors \( u, v \in V \), and let \( \text{aff}(F)^- \subseteq \text{aff}(G) \) denote the closed halfspace associated to \( \text{aff}(F) \) within \( \text{aff}(G) \).

**Definition 1.7.** Let \( P \subseteq V \) be a polytope, and define \( \mathcal{F}_k(P) \) as the set of \( k \)-dimensional faces of \( P \). Let \( G \in \mathcal{F}_{k+1}(P) \), and let \( F \) be a \( k \)-dimensional face of \( G \). The vector \( v \in \text{aff}(G) \) is a **unit outer normal vector** of \( F \) if it is a unit normal vector to \( F \), and for all \( x \in F \),

\[
\text{aff}(F)^- \cap [x, x + v] = x.
\]

![Diagram of unit outer normal vectors of faces of a cube, translated to their respective faces.](image)

Figure 1.1: Depictions of unit outer normal vectors of faces of a cube, translated to their respective faces.

The other missing piece is volume, which we define here.

**Definition 1.8.** Let \( V \) be a real, finite-dimensional vector space. A **\( d \)-dimensional volume function** on \( V \) is a map \( \text{Vol}_d : \{ P \subset V : P \text{ a } d\text{-dimensional polytope} \} \to \mathbb{R}_{\geq 0} \) with the following properties for polytopes \( P, Q \):

1. \( \text{Vol}_d(P) > 0 \) if \( \dim(P) = d \) and \( \text{Vol}_d(P) = 0 \) if \( \dim(P) < d \) (nonempty \( d \)-dimensional polytopes have nonnegative volume),
2. \( \text{Vol}_d(P + v) = \text{Vol}_d(P) \) for any \( v \in V \) (volume is preserved under translation),

3. \( \text{Vol}_d(P \cup Q) = \text{Vol}_d(P) + \text{Vol}_d(Q) - \text{Vol}_d(P \cap Q) \) whenever \( P \cup Q \) is a polytope (volume respects inclusion), and

4. If \( T : V \to V \) is a linear map, then \( \text{Vol}_d(T(P)) = |\det(T)| \text{Vol}_d(P) \) (volume respects linear maps).

A \( d \)-dimensional volume function can be defined by choosing a \( d \)-simplex and assigning it unit volume. That simplex is then used as a ‘ruler’ for \( d \)-dimensional volume. One can find an affine map \( A \) from our chosen simplex to any other \( d \)-dimensional simplex in \( V \), and under properties 2 and 4, its volume is given by the determinant of the linear component of \( A \). Since any polytope can be triangulated [2], the volume of a \( d \)-dimensional polytope can be found by computing the volume of the components of any of its triangulations.

Let \( B \) be any orthonormal basis of \( V \), and let \( \Delta \) be the simplex whose vertex set is \( B \cup \{ \vec{0} \} \). Define a \( d \)-dimensional volume function via assigning unit volume to \( \Delta \). Using this volume function, we can state Minkowski’s classical theorem.

**Theorem 1.1** (Minkowski). Let \( P \subseteq V \) be a polytope. Fix \( G \in \mathcal{F}_{k+1}(P) \), and let \( F_1, \ldots, F_n \) be its \( k \)-dimensional faces. Let \( v_i \in \text{aff}(G)' \) be the unit outer normal vector of \( F_i \). Then

\[
\sum_{i=1}^{n} \text{Vol}_k(F_i)v_i = \vec{0}.
\]

Proving Minkowski’s theorem utilizes how volume changes under projection. Before the proof, we should establish the key property. Since any face of \( P \) is full-dimensional within its
affine hull, we only need to prove that the volume of the facets of \( P \) balance their respective unit outer normal vectors.

Given two linear hyperplanes \( H_1, H_2 \subseteq V \), the intersection \( H_1 \cap H_2 \) is a codimension 2 linear subspace of \( V \). Select an orthonormal basis \( \{u_1, \ldots, u_{\dim(V)-1}\} \) of \( H_1 \) such that \( \{u_2, \ldots, u_{\dim(V)-1}\} \) is an orthonormal basis of \( H_1 \cap H_2 \). Then under projection of \( H_1 \) onto \( H_2 \), \( u_2, \ldots, u_{\dim(V)-1} \) remain unchanged, whereas \( u_1 \) is scaled by some factor. Then the unit simplex with vertices \( \{\vec{0}, u_1, \ldots, u_{\dim(V)-1}\} \) is scaled by that same factor, hence the volume of any polytope in \( H_1 \) is scaled by that same factor. So what is that factor?

Figure 1.2: A 2-dimensional example, showing the unit simplex in \( H_1 \) being compressed under projection onto \( H_2 \).

Let \( v_{H_1} \) and \( v_{H_2} \) be unit normal vectors of \( H_1 \) and \( H_2 \), respectively. Using familiar properties of orthogonal complements, we have \((H_1 \cap H_2)^\perp = \text{span}(v_{H_1}, v_{H_2})\). Recall that \( \{u_1, \ldots, u_{\dim(V)-1}\} \) is an orthonormal basis of \( H_1 \) such that \( \{u_2, \ldots, u_{\dim(V)-1}\} \) is an orthonormal basis of \( H_1 \cap H_2 \) and \( u_1 \in H_1 \). Then we know that \( u_1 \in \text{span}(v_{H_1}, v_{H_2}) \). Let \( \tilde{u}_1 \in H_2 \) such that \( \{\tilde{u}_1, u_2, \ldots, u_{\dim(V)-1}\} \) is an orthonormal basis of \( H_2 \). Either \( u_1 \ast \tilde{u}_1 \geq 0 \) or \( u_1 \ast \tilde{u}_1 < 0 \).
Without loss of generality, suppose the former. Then \( \text{proj}_{\tilde{u}_1}u_1 = (u_1 \ast \tilde{u}_1)\tilde{u}_1 \), since \( u_1 \) and \( \tilde{u}_1 \) are unit vectors. By construction, we have \( u_1 \ast v_{H_1} = 0 \) and \( \tilde{u}_1 \ast v_{H_2} = 0 \). Since \( v_{H_1} \) and \( v_{H_2} \) are also unit vectors, we have \( u_1 \ast \tilde{u}_1 = v_{H_1} \ast v_{H_2} \) if \( v_{H_1} \ast v_{H_2} \geq 0 \), and \( u_1 \ast \tilde{u}_1 = -(v_{H_1} \ast v_{H_2}) \) if \( v_{H_1} \ast v_{H_2} < 0 \).

Figure 1.3: Left: \( u_1 \ast \tilde{u}_1 = v \ast u \). Right: \( u_1 \ast \tilde{u}_1 = -(v \ast u) \).

This means that, under projection onto \( H_2 \), sets in \( H_1 \) are only scaled in the direction of \( \tilde{u}_1 \). In particular, the unit simplex with vertices \( \{ \tilde{0}, u_1, \ldots, u_{\dim(V) - 1} \} \) is compressed by a factor of \( u_1 \ast \tilde{u}_1 \) under projection onto \( H_2 \). Let \( P \subseteq V \) be a polytope, let \( F \) be a facet with a unit outer normal vector \( v \), and let \( u \) be some arbitrary unit vector. Let \( H_1 = \text{aff}(F)' \), and let \( H_2 = \text{span}(u)^\perp \). Since unit simplices are the unit of volume under \( \text{Vol}_{d-1}(\cdot) \), we have

\[
\text{Vol}_{d-1}(\tilde{F}) = \text{Vol}_{d-1}(F)(u_1 \ast \tilde{u}_1) = \pm \text{Vol}_{d-1}(F)(u \ast v).
\]

Furthermore, if \( u \ast v \geq 0 \), then

\[
\text{Vol}_{d-1}(\tilde{F}) = \text{Vol}_{d-1}(F)(u \ast v),
\]
and if $u * v < 0$, then
\[ \text{Vol}_{d-1}(\tilde{F}) = -\text{Vol}_{d-1}(F)(u * v). \]

This is the key fact we use in the proof of Minkowski’s theorem.

**Proof.** The outline of this proof is given in Rolf Schneider’s text on convex bodies [8]. Let $u \in V$ be some arbitrary unit vector, and consider the hyperplane $\text{span}(u)$. Let $\tilde{F}_i$ and $\tilde{P}$ denote the projection of each facet $F_i$ and the polytope $P$ onto $\text{span}(u)$. Notice that for each facet $\tilde{F}_i$, either $v_i * u \geq 0$ or $v_i * u < 0$. This categorizes the facets into the ‘top half’ and the ‘bottom half’ of $P$. Categorizing the facets this way, we have

\[ \tilde{P} = \bigcup_{v_i * u \geq 0} \tilde{F}_i = \bigcup_{v_i * u < 0} \tilde{F}_i. \]

Then we have
\[ \text{Vol}_{d-1}(\tilde{P}) = \sum_{v_i * u \geq 0} \text{Vol}_{d-1}(\tilde{F}_i) = \sum_{v_i * u < 0} \text{Vol}_{d-1}(\tilde{F}_i). \]

From the previous construction, we can rewrite this equation as
\[ \text{Vol}_{d-1}(\tilde{P}) = \sum_{v_i * u \geq 0} \text{Vol}_{d-1}(F_i)(v_i * u) = -\sum_{v_i * u < 0} \text{Vol}_{d-1}(F_i)(v_i * u). \]

Cancelling the right-hand side of the equation, we get
\[ \sum_{i=1}^{n} \text{Vol}_{d-1}(F_i)(v_i * u) = 0. \]

Using this result, and distributing the inner product over the sum, notice that
\[ \left( \sum_{i=1}^{n} \text{Vol}_{d-1}(F_i) v_i \right) * u = \sum_{i=1}^{n} \text{Vol}_{d-1}(F_i)(v_i * u) = 0. \]

Since $v$ is arbitrary, it must be the case that $\sum_{i=1}^{n} \text{Vol}_{d-1}(F_i)v_i = \vec{0}$, as desired. \qed
This classical theorem is a statement about polytopes, but in the simple case, it translates naturally to their normal fans (with the unit outer normal vectors being replaced by ray generators of the fan). Since normal fans of polytopes are complete, this theorem only translates to complete fans, and our goal is to re-contextualize this theorem to incomplete fans. As first steps toward a more general theorem, the main theorem of the paper will pertain to a special class of simplicial fans. To achieve this general theorem, we first need to familiarize ourselves with fans.

1.2 Fans

If our goal is to reinterpret Minkowski’s theorem to a polytopal structure whose normal fan is incomplete, we need said polytopal structure, we need fans, and we need a way to relate the two. In this section, we get the desired objects and their relevant notation, starting with the aforementioned ‘polytopal structure.’

Definition 1.9. A polyhedral complex \( \mathcal{C} \subset V \) is a finite collection of polyhedra such that

(a) \( \emptyset \in \mathcal{C} \).

(b) If a polyhedron \( P \) is in \( \mathcal{C} \), then the faces of \( P \) are in \( \mathcal{C} \).

(c) If polyhedra \( P \) and \( Q \) are in \( \mathcal{C} \), then \( P \cap Q \in \mathcal{C} \).

A polytopal complex is a polyhedral complex such that every polyhedron is bounded, i.e., a collection of polytopes satisfying the above three properties.
For example, consider the following polytopal complex comprised of 12 squares and their faces.

This particular example happens to be a normal complex, which is an instance of the species of object we need to generalize Minkowski’s theorem to incomplete fans. We define normal complexes further down the road, but first we need a few more notions.

The other species of object, ‘fans,’ have been mentioned a couple of times, but have not been properly defined. The building blocks of a fan are cones, which we give a definition for.

**Definition 1.10.** Given a nonempty set of vectors $S = \{v_1, \ldots, v_n\} \subseteq V$, the cone $\sigma \in V$ generated by $S$ is the polyhedron comprised of all linear combinations of vectors in $S$ with nonnegative coefficients:

$$\sigma = \text{cone}(S) = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \geq 0, i = 1, \ldots, n\}.$$  

Let $V_{\sigma}$ denote the linear span of a cone $\sigma$. The dimension of $\sigma$ is the dimension of its linear span: $\dim(\sigma) = \dim(V_{\sigma})$. A 1-dimensional cone is called a ray.

A fan $\Sigma \subseteq V$ is a polyhedral complex comprised only of cones. If

$$\bigcup_{\sigma \in \Sigma} \sigma = V,$$

...
we call the fan complete. Otherwise, the fan is incomplete. Given a fan $\Sigma$, let $\Sigma(k)$ denote its $k$-dimensional cones, and given a cone $\sigma$, let $\sigma(k)$ denote its $k$-dimensional faces.

By this definition, it is not obvious that cones are polyhedra (i.e., a finite intersection of closed halfspaces). However, cones are indeed polyhedra, hence fans are indeed polyhedral complexes. In particular, as shown by Ziegler [9], a cone with this definition is the finite intersection of linear halfspaces.

Incomplete fans are the fans we want to study in the end. The issue at hand is that there are a lot of types of fans, so as first steps toward extending Minkowski’s theorem, we take a look at a certain, ‘nice’ type of fan. Given a cone $\sigma$ and its generating set $S$, any face of $\sigma$ can be found by taking the cone over some subset of $S$. However, if $S$ is linearly dependent, not every subset of $S$ gives a face of $\sigma$. Otherwise, any subset of $S$ gives a face of $\sigma$. Fans whose cones have the latter property are the fans we restrict to for this project, since the relevant polytopal complexes that helps us restate Minkowski’s theorem require them. Here, we define the appropriate fans for that purpose, and give them some added structure.

**Definition 1.11.** A fan $\Sigma$ is simplicial if $\dim(V_\sigma) = |\sigma(1)|$ for all $\sigma \in \Sigma$, pure of dimension $d$ if every cone $\sigma \in \Sigma$ is a face of a cone of dimension $d$, and marked if we distinguish a nonzero vector $u_\rho$ along each ray (1-dimensional cone) $\rho$ (i.e., the ray generators of $\Sigma$ are specified).

If a fan $\Sigma$ is marked and pure of dimension $d$, we call $\Sigma$ a $d$-fan for conciseness. Simplicial $d$-fans are a class of fans we need to define normal complexes, our special polytopal complexes
that allow us to state Minkowski’s theorem for incomplete fans. Now recall that Minkowski’s theorem as stated previously only pertains to the unit outer normal vectors of its faces. We need some way to relate polytopes to fans. We do this via the following definition.

**Definition 1.12.** Let \( P \subseteq V \) be a full-dimensional polytope, and let \( F \) be a face of \( P \). Let \( u_G \) denote an outer normal vector of the facet \( G \) of \( P \). The **normal cone** \( \sigma_F \) associated to \( F \) is the cone

\[
\sigma_F := \text{cone} \left( u_G : G \text{ a facet of } P \text{ containing } F \right),
\]

and the **normal fan** \( \Sigma_P \) is the fan comprised of each \( \sigma_F \) for every face \( F \subseteq P \setminus \emptyset \). If \( \Sigma_P \) is simplicial, then we call \( P \) **simple**.

Each facet of \( P \) corresponds to a ray of \( \Sigma_P \), each \( d-2 \) dimensional face of \( P \) corresponds to a 2-dimensional cone of \( \Sigma_P \), and so on [9]. The dimension of the faces of \( P \) correspond inversely to the dimension of the cones in \( \Sigma_P \). Notice that the unit outer normal vectors of the facets \( F \) of \( P \) are precisely the unit-length ray generators \( u_{\rho_F} \) of \( \Sigma_P \). Now we can think about rephrasing Minkowski’s theorem in terms of the normal fan of a polytope.

Rephrasing Minkowski’s theorem for the facets of a polytope \( P \subseteq V \) is immediate, since the unit outer normal vectors of the facets of \( P \) are ray generators of \( \Sigma_P \). The more interesting construction is the generalization of Minkowski’s theorem to all dimensions of cones of \( \Sigma_P \). For this construction, we will assume that \( P \subseteq V \) is a simple \( d \)-dimensional polytope, since the central object of this project has a simplicial normal fan. We establish some notation about simplicial fans.
Let $\Sigma$ be a simplicial fan. If $\tau$ is a face of a cone $\sigma \in \Sigma$, we denote the containment $\tau \preceq \sigma$. For proper containment, we denote it $\tau \prec \sigma$. If $\tau \preceq \sigma$, denote by $\sigma \setminus \tau$ the face of $\sigma$ containing the rays $\sigma(1) \setminus \tau(1)$.

![Figure 1.4: A 3-dimensional cone $\sigma$ with a highlighted faces $\tau$ and $\sigma \setminus \tau$.](image)

Using this, we highlight the case that $\tau$ is a facet of $\sigma$. In that case, $|\tau(1)| = |\sigma(1)| - 1$, hence $\sigma \setminus \tau$ is a ray.

Let $P \subseteq V$ be a simple $d$-dimensional polytope. Fixing a face in $P$ is equivalent to fixing a cone in $\Sigma_P$, so fix $\tau \in \Sigma_P(k)$, and denote its corresponding face of $P$ by $F_\tau \in \mathcal{F}_{d-k}(P)$. For all cones $\sigma \in \Sigma_P(k + 1)$ containing $\tau$ as a face, we get each face $F_\sigma \in \mathcal{F}_{d-k-1}(P)$ that are faces of $F_\tau$. Then for any such $F_\sigma$, $\tau$ is a facet of $\sigma$, so $\sigma \setminus \tau$ is a ray in $\Sigma_P$. Let $u_{\sigma \setminus \tau}$ denote the ray generator of $\sigma \setminus \tau$. By Minkowski's theorem, we know that the volume of each $F_\sigma$ gives a balancing condition on their respective unit outer normal vectors $v_{F_\sigma} \in \text{aff}(F_\tau)$.

In translating Minkowski’s theorem to $\Sigma_P$, we want the balancing condition to be on the ray generators $u_{\sigma \setminus \tau}$. The hangup here is that those ray generators may need to be scaled, since their projection onto $\text{aff}(F_\tau)$ may not be unit length. We can scale the ray generators
accordingly via inspiration from Fulton and Sturmfels in [3]. For a cone \( \sigma \in \Sigma_P(k + 1) \) and its ray generators \( u_\rho \), let \( [\vec{0}, u_\rho] \) denote the line segment from \( \vec{0} \) to \( u_\rho \). Then define

\[
\mathcal{P}_\sigma := \sum_{\sigma \in \Sigma_P(k) \atop \rho \in \sigma(1)} [\vec{0}, u_\rho] \subseteq \sigma,
\]

where the sum is a Minkowski sum. That is, \( \mathcal{P}_\sigma \) is the parallelepiped generated by the ray generators of \( \sigma \). Fix a cone \( \tau \in \Sigma_P(k) \). For any cone \( \Sigma_P(k + 1) \) such that \( \tau \) is a face of \( \sigma \), we have \( \sigma \setminus \tau \) as a ray of \( \sigma \) not contained in \( \tau \). Choose a ray generator \( u_{\sigma \setminus \tau} \) of \( \sigma \setminus \tau \), and let \( v_{F_\sigma} \in \text{aff}(F_\tau)' \) be the unit outer normal vector of the face \( F_\sigma \) of \( P \) associated to the cone \( \sigma \in \Sigma_P \). Note that the volume of a parallelepiped can be computed via the volume of its facet, multiplied with its ‘height’ in the direction of a normal vector to that facet. In this case, we have \( \mathcal{P}_{\tau} \) as a facet of \( \mathcal{P}_\sigma \), and the ‘height’ is \( \| \text{proj}_{v_{F_\tau}} u_{\sigma \setminus \tau} \| \), since \( v_{F_\sigma} \) is normal to \( V_\tau \). Then we have

\[
\frac{1}{\text{Vol}_{k+1}(\mathcal{P}_\sigma)} u_{\sigma \setminus \tau} = \frac{1}{\text{Vol}_k(\mathcal{P}_\tau) \| \text{proj}_{v_{F_\tau}} u_{\sigma \setminus \tau} \|} u_{\sigma \setminus \tau}.
\]

Note that \( \text{proj}_{v_{F_\sigma}} u_{\sigma \setminus \tau} \) may not be a unit vector. However, we do have

\[
\text{proj}_{v_{F_\sigma}} u_{\sigma \setminus \tau} = \| \text{proj}_{v_{F_\sigma}} u_{\sigma \setminus \tau} \| v_{F_\sigma}.
\]

Then let

\[
\tilde{u}_{\sigma \setminus \tau} = \frac{1}{\| \text{proj}_{v_{F_\sigma}} u_{\sigma \setminus \tau} \|} u_{\sigma \setminus \tau},
\]

so \( \text{proj}_{v_{F_\sigma}} \tilde{u}_{\sigma \setminus \tau} = v_{F_\sigma} \). Then we have

\[
\frac{1}{\text{Vol}_k(\mathcal{P}_\tau) \| \text{proj}_{v_{F_\sigma}} u_{\sigma \setminus \tau} \|} u_{\sigma \setminus \tau} = \frac{1}{\text{Vol}_k(\mathcal{P}_\tau)} \tilde{u}_{\sigma \setminus \tau}.
\]
The parallelepiped $P_\tau$ is the same regardless of the cone $\sigma \in \Sigma(k + 1)$ that $\tau$ is a face of, so
\[
\sum_{\sigma \in \Sigma_P(k+1) \atop \tau \preceq \sigma} \frac{\text{Vol}_{d-k-1}(F_\sigma)}{\text{Vol}_{k+1}(P_\sigma)} u_{\sigma \setminus \tau} = \frac{1}{\text{Vol}_k(P_\tau)} \sum_{\sigma \in \Sigma_P(k+1) \atop \sigma \prec \tau} \text{Vol}_{d-k-1}(F_\sigma) \tilde{u}_{\sigma \setminus \tau},
\]
To better understand what this construction accomplishes, we should take a look at an example.

**Example 1.2.** Let $P$ be the following simplex in $\mathbb{R}^3$. Take the ray generators of $\Sigma_P$ to be the unit outer normal vectors of $P$. In this example, we verify Minkowski’s theorem on the highlighted facet $F$:

![Figure 1.5: The simplex $P$ with unit outer normal vectors translated to their respective facets.](image)

To verify Minkowski’s theorem on $F$, we use the standard volumes of the three 1-dimensional faces of $F$. Denote those faces by
\[
e_1 = [(2, 0, 0), (0, 0, 2)], \quad e_2 = [(0, 0, 0), (2, 0, 0)], \quad \text{and} \quad e_3 = [(0, 0, 0), (0, 0, 2)].
\]
Using the standard distance formula in $\mathbb{R}^3$, we have
\[
\text{Vol}_1(e_1) = 2\sqrt{2}, \quad \text{Vol}_1(e_2) = 2, \quad \text{and} \quad \text{Vol}_1(e_3) = 2.
\]
CHAPTER 1. SHAPES

The cone corresponding to $F$ is $\rho = \text{cone}(v_4)$, and the ray generators corresponding to the unit outer normal vectors of $e_1, e_2, \text{ and } e_3$ are $v_1, v_2, \text{ and } v_3$ respectively. We want a weighted sum of $v_1, v_2, \text{ and } v_3$ to be in the span of $\rho$. Simply weighting $v_1, v_2, \text{ and } v_3$ with the 1-dimensional volumes of $e_1, e_2, \text{ and } e_3$ does not work for us:

$$
\text{Vol}_1(e_1)v_1 + \text{Vol}_1(e_2)v_2 + \text{Vol}_1(e_3)v_3 = 2\sqrt{2} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + 2(0,0,-1) + 2(-1,0,0)
$$

$$
= \left( \frac{2\sqrt{2}}{\sqrt{3}} - 2, \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}} - 2 \right) \notin \text{span}(\rho).
$$

The orthogonal projection of $v_i$ onto the unit outer normal vector of $e_i$ (parallel to the affine span of $F$) is not unit length, which is why the weighted sum is not in $\text{span}(\rho)$. To account for this, we scale the ray generators using parallelepipeds.

Let $\mathcal{P}_i$ denote the parallelepiped generated by $v_4$ and $v_i$, where $i = 1, 2, 3$. Then we have

$$
\text{Vol}_2(\mathcal{P}_1) = \frac{\sqrt{6}}{3}, \text{ Vol}_2(\mathcal{P}_2) = 1, \text{ and } \text{Vol}_2(\mathcal{P}_3) = 1.
$$

As in the construction, dividing the volumes of each $e_i$ by the volume of $\mathcal{P}_i$ should be the appropriate weight for the weighted sum to be in $\text{span}(\rho)$. We verify this:

$$
\begin{align*}
\frac{\text{Vol}_1(e_1)}{\text{Vol}_2(\mathcal{P}_1)}v_1 + \frac{\text{Vol}_1(e_2)}{\text{Vol}_2(\mathcal{P}_2)}v_2 + \frac{\text{Vol}_1(e_3)}{\text{Vol}_2(\mathcal{P}_3)}v_3 \\
= 2\sqrt{2} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + \frac{2}{1}(0,0,-1) + \frac{2}{1}(-1,0,0) \\
= 2\sqrt{3} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) + 2(0,0,-1) + 2(-1,0,0) \\
= (2,2,2) + (0,0,-2) + (-2,0,0) \\
= (0,2,0) \in \text{span}(\rho).
\end{align*}
$$
The construction allows us to state Minkowski’s theorem in terms of normal fans, in the style of Fulton and Sturmfels [3]

**Theorem 1.2** (Minkowski). Let $P \subseteq V$ be a simple $d$-dimensional polytope with normal fan $\Sigma_P$. For all $\tau \in \Sigma_P(k)$, we have

$$\sum_{\sigma \in \Sigma_P^{k+1} \cap \tau} \frac{\text{Vol}_{d-k-1}(F_{\sigma})}{\text{Vol}_{k+1}(P_{\sigma})} u_{\sigma \cap \tau} \in V_{\tau}.$$ 

This theorem utilizes the relationship between the faces of a polytope and the cones of its normal fan. In order to re-work this in the context of incomplete fans, we need some polytopal object whose “normal fan” is incomplete. To do this, instead of starting from the polytope, we start from its normal fan.

Let $P \subseteq V$ be a polytope with normal fan $\Sigma_P$. In reinterpreting Minkowski’s classical theorem to normal fans, notice that facet volume intuitively ‘weights’ the rays of $\Sigma_P$ in a sense. This idea gives rise to a more general notion regarding fans.

**Definition 1.13.** Let $\Sigma$ be a $d$-fan. A $k$-**weight** on $\Sigma$ is a function $\omega : \Sigma(k) \to \mathbb{R}$.

In Minkowski’s theorem, by this definition, the volume function $\text{Vol}_{d-k}$ is a $k$-weight on $\Sigma_P$ that sends each $k$-dimensional cone of $\Sigma_P$ to the volume of its associated $(d - k)$-dimensional face. This particular $k$-weight is special, in that it satisfies our desired balancing condition, i.e., that the weighted sum of the ray generators neighboring a given cone of $\Sigma_P$ lands in the linear span of that cone. This $k$-weight is an example of a broader type of $k$-weight that balances $k$-dimensional cones that share a given $(k - 1)$-dimensional cone [3].
Definition 1.14. Let $\Sigma$ be a simplicial $d$-fan. A $k$-weight $\omega$ is a **Minkowski weight** if

$$\sum_{\sigma \in \Sigma(k) \atop \sigma \succ \tau} \omega(\sigma) u_{\sigma \setminus \tau} \in V_{\tau} \quad \text{for all} \quad \tau \in \Sigma(k - 1).$$

If $\Sigma$ has a Minkowski $d$-weight, then we call the pair $(\Sigma, \omega)$ a **tropical fan**. If $(\Sigma, \omega)$ is a tropical fan such that $\omega(\sigma) = 1$ for all $\sigma \in \Sigma(d)$, then we say that $\Sigma$ is **balanced**, and we drop $\omega$ from the notation.

Any complete $d$-fan is tropical with an appropriate $d$-weight, so the normal fan $\Sigma_P$ of a $d$-dimensional polytope $P$ is a tropical fan. Furthermore, Minkowski’s theorem says that the volumes of $(d - k)$-dimensional faces of a polytope $P$ give a Minkowski $k$-weight on $\Sigma_P$. Recall that our goal is to extend Minkowski’s theorem to incomplete fans... What does an incomplete tropical fan look like?

When looking at new geometric ideas, we like to have a familiar face to keep us grounded. Consider the following example. Let $V = \mathbb{R}^3$, and let $u_1, u_2, u_3$ be the standard basis vectors of $\mathbb{R}^3$. Let $u_0 = -(u_1 + u_2 + u_3)$, and for any subset $S \subseteq \{0, 1, 2, 3\}$, let $u_S = \sum_{i \in S} u_i$. Let $\rho_S$ be the ray spanned by $u_S$. Below is a balanced 2-fan whose generators are of the form described.
CHAPTER 1. SHAPES

1.3 Normal Complexes

Now that we have the appropriate structure surrounding fans, we need to work back to polytopal complexes. In this section, we cook up the relevant polytopal complex — the normal complex. Our ingredients are as follows: a simplicial $d$-fan $\Sigma \subseteq V$, an inner product $* \in \text{Inn}(V)$, and a vector $z \in \mathbb{R}^{\Sigma(1)}$. Given these ingredients, we get sets of halfspaces and hyperplanes associated to each ray $\rho \in \Sigma(1)$. Define

$$H_{\rho,*}(z) := \{ v \in V : v * u_\rho = z_\rho \},$$

the hyperplane associated to $\rho$, and

$$H_{\rho,*}(z) := \{ v \in V : v * u_\rho \leq z_\rho \},$$

Figure 1.6: We will refer back to this balanced fan a few times to build further concepts.
the halfspace associated to $\rho$. Then define polytopes

$$P_{\sigma, *}(z) := \sigma \cap \bigcap_{\rho \in \sigma(1)} H^{-}_{\rho, *}(z)$$

for each cone $\sigma \in \Sigma$. To see this definition in action, consider the following example.

**Example 1.3.** Consider the following 2-dimensional cone $\sigma$ containing rays $\rho_1$ and $\rho_2$ with ray generators $u_{\rho_1} = (3, 1)$ and $u_{\rho_2} = (1, 3)$.

![Diagram](image1)

Given $z \in \mathbb{R}^{\sigma(1)}$ and an inner product, we construct halfspaces corresponding to $\rho_1$ and $\rho_2$.

![Diagram](image2)

Figure 1.7: Notice that $H_{\rho_1, *}(z)$ and $H_{\rho_2, *}(z)$ intersect within $\sigma$. 
In this case, we have $z = (9,9)$. Intersecting $\sigma$ with $H^{-1}_{\rho_1,*}(z)$ and $H^{-1}_{\rho_2,*}(z)$ gives us the polytope $P_{\sigma,*}(z)$, as shown below.

Our choice of $z$-values in example (1.3) is such that $H^{-1}_{\rho_1,*}(z)$ and $H^{-1}_{\rho_2,*}(z)$ intersect on the interior of $\sigma$, giving us a quadrilateral. Depending on our choice of $z$, this may not be the case. To formalize this, for each $\sigma \in \Sigma$, let $w_{\sigma,*}(z) \in V_\sigma$ be the unique vector such that $w_{\sigma,*}(z) \cdot u_{\rho} = z_{\rho}$ for all $\rho \in \sigma(1)$. This vector is the point at which each of the hyperplanes $H_{\rho,*}(z)$ intersect, and we know they intersect at a single point since $\sigma$ is a simplicial cone.

For a polyhedron $P \subseteq V$, denote by $P^\circ$ the relative interior of $P$. That is, let $\{H_i\}_{i \in I}$ denote the set of supporting hyperplanes of $P$ as a full-dimensional polytope within $\text{aff}(P)$. Then

$$P^\circ := \{x \in P : x \notin H_i \ \forall \ i \in I\}.$$ 

Using this, consider the following.

**Definition 1.15.** We say that $z \in \mathbb{R}^{\Sigma(1)}$ is cubical with respect to $(\Sigma, \ast)$ if $w_{\sigma,*}(z) \in \sigma^\circ$ for all $\sigma \in \Sigma$, and we say that $z$ is pseudocubical with respect to $(\Sigma, \ast)$ if $w_{\sigma,*}(z) \in \sigma$ for all $\sigma \in \Sigma$. 
This means that the hyperplanes corresponding to cubical and pseudocubical $z$-values intersect in their respective cones. For example, the $z$ chosen as in (1.3) was cubical. The following figure shows a choice of $z \in \mathbb{R}^{\sigma(1)}$ that is neither cubical nor pseudocubical.

![Figure 1.8](image)

Figure 1.8: Here, $u_{\rho_1} = (1, 3)$, $u_{\rho_2} = (3, 1)$, and $z = (6, 2)$.

For a simplicial $d$-fan $\Sigma$ and inner product $\ast \in \text{Inn}(V)$, denote $\text{Cub}(\Sigma, \ast) \subseteq \mathbb{R}^{\Sigma(1)}$ as the set of cubical values, and $\overline{\text{Cub}}(\Sigma, \ast) \subseteq \mathbb{R}^{\Sigma(1)}$ the set of pseudocubical values.

**Definition 1.16.** Given a simplicial $d$-fan $\Sigma \subseteq V$, an inner product $\ast \in \text{Inn}(V)$, and $z \in \overline{\text{Cub}}(\Sigma, \ast)$, the **normal complex of $\Sigma$ with respect to $\ast$ and $z$** is the polytopal complex

$$C_{\Sigma, \ast}(z) := \bigcup_{\sigma \in \Sigma(d)} \overline{P_{\sigma, \ast}(z)}.$$

The fact that normal complexes are well-defined polytopal complexes can be attributed to Nathanson and Ross [6].

**Example 1.4.** For example, recall the following 2-dimensional balanced fan $\Sigma \subseteq \mathbb{R}^3$. 
By letting $\ast$ denote the dot product on $\mathbb{R}^3$ and choosing $z = (2, 2, 2, 3, 3, 3, 3, 3, 3) \in \text{Cub}(\Sigma, \ast)$, we get the following polytopal complex, whose nine quadrilaterals correspond to the nine 2-dimensional cones in the previous fan. This is a normal complex with respect to $\Sigma$.

Figure 1.9: This normal complex is informally called a “fletching,” which is the feathery back-end of a dart/arrow. The name was coined by Lauren Nowak.
1.4 Volumes of Normal Complexes

This project revolves around computing the volume of the faces of normal complexes. Typically, when we think about volume (area, length, etc.), we have some kind of pre-established metric to compute it for any object in the space, usually given to us by an inner product. In our case, some care needs to be taken when computing volumes of normal complexes, since its elements are subsets of different subspaces of $V$.

Let $\Sigma \subseteq V$ be a simplicial $d$-fan, * an inner product on $V$, and $z \in \text{Cub}(\Sigma)$. For each $\sigma \in \Sigma(d)$, define a basis $\{v_\eta\}$ of $V_\sigma$ by

$$v_\eta \ast u_\mu = \begin{cases} 1, & \eta = \mu; \\ 0, & \eta \neq \mu \end{cases}$$

for all $u_\eta \in \sigma(1)$. Let $\Delta$ be the simplex whose vertices are $\vec{0}$ and the basis vectors. Then define a volume function

$$\text{Vol}_\sigma : V_\sigma \rightarrow \mathbb{R}_{\geq 0}$$

determined by the property that $\Delta \subseteq V_\sigma$ has volume 1. This gives us a unit of volume for each polytope $P_{\sigma,*}(z) \subseteq V_\sigma$. To measure the volume of the whole normal complex, we define the volume of the normal complex $C_{\Sigma,*}(z)$ as

$$\text{Vol}(C_{\Sigma,*}(z)) := \sum_{\sigma \in \Sigma(d)} \text{Vol}_\sigma(P_{\sigma,*}(z)),$$  \hspace{1cm} (1.1)

the sum of the volumes of each polytope in $C_{\Sigma,*}(z)$. Further, if $\omega : \Sigma(d) \rightarrow \mathbb{R}$ is a $d$-weight,
define the volume of $C_{\Sigma, s}(z)$ weighted by $\omega$ as

$$\text{Vol}(C_{\Sigma, s}(z); \omega) := \sum_{\sigma \in \Sigma(d)} \omega(\sigma) \text{Vol}_\sigma(P_{\sigma, s}(z)).$$

With this definition, if $\Sigma$ is a balanced fan, the weighted volume of $C_{\Sigma, s}(z)$ is given by (1.1).

**Example 1.5.** Let $\Sigma \in \mathbb{R}^2$ be a fan comprised of a single 2-dimensional cone $\sigma$ with rays $\rho$ and $\eta$. Here, for any $z \in \overline{\text{Cub}}(z)$, we have $C_{\Sigma, s}(z) = P_{\sigma, s}(z)$. Let $u_\rho = (2, 1)$ and $u_\eta = (1, 2)$. The basis of $\mathbb{R}^2$ associated to $\sigma$ is $\{v_\rho, v_\eta\}$, where $v_\rho = \left(\frac{2}{3}, \frac{-1}{3}\right)$ and $v_\eta = \left(\frac{-1}{3}, \frac{2}{3}\right)$.

The simplex $\Delta$ whose vertices are $\{\vec{0}, v_\rho, v_\eta\}$ serves as the unit of volume for $P_{\sigma, s}(z)$ for any $z \in \overline{\text{Cub}}(z)$. If we let $z = (10, 10)$, then $P_{\sigma, s}(z)$ is the following quadrilateral.

![Figure 1.10: The quadrilateral $P_{\sigma, s}(z)$, with the unit-volume simplex overlaid.](image)
In this example, we can tile $P_{\sigma,*}(z)$ with 40 copies of $\Delta$, hence $\text{Vol}_\sigma(P_{\sigma,*}(z)) = 40$. Since the only 2-dimensional cone of $\Sigma$ is $\sigma$, we have $\text{Vol}(C_{\Sigma,*}(z)) = 40$.

In a more general setting, we can equivalently compute the volume of $P_{\sigma,*}(z)$ by triangulating $P_{\sigma,*}(z)$ and finding affine transformations such that each resulting simplex is the image of $\Delta$, then summing the determinants of each transformation.

As stated before, we need the volumes of the faces of normal complexes, not the volume of the whole normal complex. This raises two questions: what is the face of a normal complex, and how do we compute its volume? In short, we assert that the answers to these questions are as follows: faces of normal complexes are also normal complexes, thus their volume should be computed in the same manner.

To give a full answer to those questions, we should first properly define what a face of a normal complex is. Here is the plan: by intersecting $C_{\Sigma,*}(z)$ with a subset of its supporting hyperplanes, we obtain some polytopal complex. Normal complexes are always centered at the origin, so we must then translate this intersection to the origin. We can assign the resulting polytopal complex the structure of a normal complex.
1.5 Faces and Face Volumes

For each cone $\tau \in \Sigma$, define

$$V^\tau = V/V_\tau.$$  

Under the inner product $\ast$, $V^\tau$ may be identified as the orthogonal complement of $V_\tau$, giving us $V = V_\tau \oplus V^\tau$. Denote the orthogonal projection onto the factors of this decomposition by $\text{pr}_\tau$ and $\text{pr}^\tau$ respectively.

Next, for each cone $\tau \in \Sigma$, define the neighborhood of $\tau$ in $\Sigma$ via

$$N_\tau \Sigma := \{ \pi : \pi \preceq \sigma \text{ for some } \sigma \in \Sigma \text{ with } \tau \preceq \sigma \} \subseteq \Sigma.$$  

That is, if a cone $\pi \in \Sigma$ is in the neighborhood of $\tau$, then there exists some cone $\sigma \in \Sigma$ that $\pi$ and $\tau$ are both faces of. The following figure highlights the neighborhood of a ray of a familiar simplicial 2-fan.

![Diagram of a simplicial 2-fan with a highlighted neighborhood of a ray]

Note that $N_\tau \Sigma$ is a simplicial $d$-fan. In particular, it is a fan in $V$ whose maximal cones are the maximal cones of $\Sigma$ containing $\tau$. Then every maximal cone $\sigma \in N_\tau \Sigma(d)$ has $\tau$ as a face, so for every $\rho \in \tau(1)$, $H_{\rho,\ast}(z)$ is a supporting hyperplane of $P_{\sigma,\ast}(z)$. 
Now, for every $\sigma \in N_\tau \Sigma(d)$, we get a face of $P_{\sigma,*}(z)$ via

$$F_\tau(P_{\sigma,*}(z)) := P_{\sigma,*}(z) \cap \bigcap_{\rho \in \tau(1)} H_{\rho,*}(z).$$

The collection of these polytopes gives us a polytopal subcomplex of $C_{\Sigma,*}(z)$:

$$F_\tau(C_{\Sigma,*}(z)) := \bigcup_{\sigma \in N_\tau \Sigma(d)} F_\tau(P_{\sigma,*}(z)).$$

Figure 1.11: The red facet is the intersection of the normal complex $C_{\sigma,*}(z)$ with one of its supporting hyperplanes $H_{\rho,*}(z)$. 
Note that $F_\tau(C_{\Sigma,*}(z))$ is a face of $C_{\Sigma,*}(z)$ in the sense of Definition 1.6. However, our goal is to talk about faces of $C_{\Sigma,*}(z)$ as normal complexes. Normal complexes always contains the origin, and $F_\tau(C_{\Sigma,*}(z))$ does not necessarily contain the origin. Our next step is to account for that by translating $F_\tau(C_{\Sigma,*}(z))$. In particular, notice that

$$\bigcap_{\rho \in \tau(1)} H_{\rho,*}(z) = V^\tau + w_{\tau,*}(z).$$

By definition,

$$F_\tau(C_{\Sigma,*}(z)) \subset \bigcap_{\rho \in \tau(1)} H_{\rho,*}(z) = V^\tau + w_{\tau,*}(z),$$

so translating $F_\tau(C_{\Sigma,*}(z))$ by $-w_{\tau,*}(z)$ gives a polytopal complex in $V^\tau$. Define the face of $C_{\Sigma,*}(z)$ associated to $\tau \in \Sigma$ by

$$F^\tau(C_{\Sigma,*}(z)) := F_\tau(C_{\Sigma,*}(z)) - w_{\tau,*}(z) \subseteq V^\tau.$$

It is shown by Nowak, O’Melveny, and Ross in [7] that $F^\tau(C_{\Sigma,*}(z))$ is a normal complex. In particular,

$$F^\tau(C_{\Sigma,*}(z)) = C_{\Sigma^\tau,*^\tau}(z^\tau),$$

where $\Sigma^\tau$ is the projection of $N_\tau \Sigma$ onto $V^\tau$, $*^\tau \in \text{Inn}(V^\tau)$ is the restriction of $*$ to $V^\tau$, and $z^\tau$ is given by $z^\tau_{\rho} = z_{\hat{\rho}} - w_{\tau,*}(z) * u_{\hat{\rho}}$, $\hat{\rho}$ being the projection of a ray $\rho \in \Sigma(1)$ onto $V^\tau$.

Since the volume of a normal complex is at its core determined by the markers of its associated fan, the same applies to the volumes of the faces of normal complexes. That being said, the volume of the face $C_{\Sigma^\tau,*^\tau}(z^\tau)$ of $C_{\sigma,*}(z)$ weighted by $\omega$ is

$$\text{Vol}(F^\tau(C_{\Sigma,*}(z)); \omega).$$
Chapter 2

Minkowski’s Theorem for 2-D Tropical Fans

Let $P$ be a $d$-dimensional polytope. Recall that Minkowski’s theorem states that the volume of the $(d - k)$-dimensional faces of $P$ is a Minkowski $k$-weight on its normal fan $\Sigma_P$. This project is motivated toward a proof of a more general conjecture that re-works Minkowski’s theorem to apply to incomplete tropical fans.

**Conjecture 2.1.** Let $(\Sigma, \omega)$ be a simplicial, tropical $d$-fan in $V$, $*$ an inner product on $V$, $z \in \overline{\text{Cub}}(\Sigma, *)$, and $C_{\Sigma,*}(z)$ the normal complex of $\Sigma$ with respect to $*$ and $z$. Define a weight function

$$\omega_{\text{Vol}} : \Sigma(d - k) \to \mathbb{R}$$

$$\tau \mapsto \text{Vol}(F^\tau(C_{\Sigma,*}(z)); \omega).$$
Then \( \omega_{\text{Vol}_k} \) is a Minkowski \((d-k)\)-weight on \( \Sigma \).

In this chapter, we prove a low-dimensional case of this conjecture — that the facets of a 2-dimensional normal complex are a Minkowski 1-weight on its normal fan.

### 2.1 Facets of 2-Dimensional Normal Complexes

In this section, we construct a formula for the volume of a face of a 2-dimensional normal complex and generalize Minkowski’s theorem to incomplete 2-dimensional tropical fans. The facets of a 2-dimensional normal complex are 1-dimensional normal complexes, i.e., a complex of line segments that all share one endpoint. This means that measuring their volume amounts to measuring the length of each line segment and summing the lengths, using the associated basis as a measurement of length.

To assist in a proof of the 2-dimensional case, we construct a closed formula for the volume of a facet of a 2-dimensional normal complex. As inspiration, we compute an example.

**Example 2.1.** Let \( \Sigma \in \mathbb{R}^2 \) be a 2-dimensional fan, and let \( \sigma \in \Sigma \) be a 2-dimensional cone with rays \( \rho \) and \( \eta \). Let \( u_\rho = (2,1) \) and \( u_\eta = (1,2) \). Let \( z = (10,10) \). The basis of \( \mathbb{R}^2 \) associated to \( \sigma \) is \( \{v_\rho, v_\eta\} \), where \( v_\rho = \left(\frac{2}{3}, \frac{-1}{3}\right) \) and \( v_\eta = \left(\frac{-1}{3}, \frac{2}{3}\right) \). Let us compute the volume (i.e., the length) of the highlighted face of \( P_{\sigma,*}(z) \).
To compute the length of $F_\rho(P_{\sigma,*}(z))$, we only need to find the scalar projection of $w_{\sigma,*}(z)$ onto $v_\eta$. Here, we have

$$w_{\sigma,*}(z) = 10v_\rho + 10v_\eta.$$ 

Then we can compute the projection:

$$\text{Vol}(F_\rho(P_{\sigma,*}(z))) = \frac{w_{\sigma,*}(z) \ast v_\eta}{v_\eta \ast v_\eta} = \frac{(10v_\rho + 10v_\eta) \ast v_\eta}{v_\eta \ast v_\eta} = 10v_\rho \ast v_\eta + 10v_\eta \ast v_\eta.$$ 

This is an expression of the length of $F_\rho(P_{\sigma,*}(z))$ in terms of $v_\rho$ and $v_\eta$. The issue here is that $v_\rho$ and $v_\eta$ both change depending on the cone $\sigma$ of $\Sigma$, so perhaps we could rewrite this formula in terms of $u_\rho$ and $u_\eta$, where $u_\rho$ is consistent throughout any cone $\sigma$ of $\Sigma$ containing $\rho$. Notice:

$$\frac{u_\rho \ast u_\eta}{v_\eta \ast v_\eta} = \frac{(\frac{2}{5}, \frac{1}{3}) \ast (\frac{-1}{3}, \frac{2}{3})}{(\frac{-1}{3}, \frac{2}{3}) \ast (\frac{-1}{3}, \frac{2}{3})} = \frac{-4}{5}.$$
Notice as well that
\[
\frac{u_\eta * u_\rho}{u_\rho * u_\rho} = \frac{(1, 2) * (2, 1)}{(2, 1) * (2, 1)} = \frac{4}{5}.
\]
Then we have
\[
\frac{v_\rho * v_\eta}{v_\eta * v_\eta} = -\frac{u_\eta * u_\rho}{u_\rho * u_\rho},
\]
so
\[
\text{Vol}(F_\rho(P_{\sigma,*(z)})) = 10 - 10 \frac{u_\eta * u_\rho}{u_\rho * u_\rho} = 10 - 10 \left(\frac{4}{5}\right) = 2.
\]
In further generality, we have
\[
\text{Vol}(F_\rho(P_{\sigma,*(z)})) = z_\eta - z_\rho \frac{u_\eta * u_\rho}{u_\rho * u_\rho}.
\]
If we can show that this formula applies to any 2-dimensional polytope $P_{\sigma,*(z)}$ of a 2-dimensional normal complex $C_{\Sigma,*(z)}$, then the volume of a face of $C_{\Sigma,*(z)}$ is taken by summing that expression over the rays $\eta \in N_\rho \Sigma \setminus \{\rho\}$. In particular, we want to verify the following relationship:
\[
\frac{v_\rho * v_\eta}{v_\eta * v_\eta} = -\frac{u_\eta * u_\rho}{u_\rho * u_\rho}.
\]
Here, we show that the results of the previous example hold in general for 2-dimensional normal complexes. The figure below depicts a quadrilateral $P_{\sigma,*(z)}$ from a 2-dimensional normal complex $C_{\Sigma,*(z)}$, along with the basis $v_\rho, v_\eta$ associated to the cone $\sigma \in \Sigma$. 
Notice that the highlighted facet of $P_{\sigma, s}(z)$ is parallel to $v_\eta$. Since $v_\eta$ is a unit simplex in $V^\rho$, the scalar projection of $w_{\sigma, s}(z)$ onto $v_\eta$ gives us the length of the highlighted face. Recalling the definition of $w_{\sigma, s}(z)$, we have the system of equations

\[
\begin{align*}
   w_{\sigma, s}(z) * u_\rho &= z_\rho; \\
   w_{\sigma, s}(z) * u_\eta &= z_\eta.
\end{align*}
\]

Writing $w_{\sigma, s}(z) = av_\rho + bv_\eta$, we have

\[
\begin{align*}
   (av_\rho + bv_\eta) * u_\rho &= a v_\rho * u_\rho + b v_\eta * u_\eta = a = z_\rho; \\
   (av_\rho + bv_\eta) * u_\eta &= a v_\rho * u_\eta + a v_\eta * u_\eta = b = z_\eta.
\end{align*}
\]

So $w_{\sigma, s}(z) = z_\rho v_\rho + z_\eta v_\eta$. Then the length of the face of $P_{\sigma, s}(z)$ associated to $\rho$ is obtained by projecting $w_{\sigma, s}(z)$ onto $v_\eta$:

\[
\frac{(z_\rho v_\rho + z_\eta v_\eta) * v_\eta}{v_\eta * v_\eta} = \frac{z_\rho v_\rho * v_\eta}{v_\eta * v_\eta} + \frac{z_\eta v_\eta * v_\eta}{v_\eta * v_\eta} = z_\rho \frac{v_\rho * v_\eta}{v_\eta * v_\eta} + z_\eta.
\]

(2.1)

Notice that the first term of the right-hand side of the equation includes the scalar projection of $v_\rho$ onto $v_\eta$ as a factor. Minkowski weights are paired with the generating vectors of their associated fan, not vectors in the dual basis, so we aim to rewrite this formula in terms of $u_\rho$ and $u_\eta$. Staring at the diagram a little longer, we see that the angle between $v_\rho$ and $-v_\eta$ is equal to the angle between $u_\eta$ and $u_\rho$. This suggests a relationship between the projection of $v_\rho$ onto $v_\eta$ and the projection of $u_\eta$ onto $u_\rho$. 
For a 2-dimensional cone $\sigma$ with rays $\rho$ and $\eta$, let $u_\rho = (a_\rho, b_\rho)$ and $u_\eta = (a_\eta, b_\eta)$ with respect to any basis of $V_\sigma$. Then let $\bar{v}_\rho = (b_\eta, -a_\eta)$ and $\bar{v}_\eta = (-b_\rho, a_\rho)$. By construction, $\bar{v}_\rho$ and $\bar{v}_\eta$ are $v_\rho$ and $v_\eta$, respectively, scaled by some factor. Furthermore, $u_\rho \ast u_\rho = \bar{v}_\eta \ast \bar{v}_\eta$ and $u_\eta \ast u_\eta = \bar{v}_\rho \ast \bar{v}_\rho$. This means that $u_\rho$ has the same length as $\bar{v}_\eta$ and $u_\eta$ has the same length as $\bar{v}_\rho$. 
Notice that

\[ u_\rho \ast \bar{v}_\rho = a_\rho b_\eta - b_\rho a_\eta; \]

\[ u_\eta \ast \bar{v}_\eta = a_\rho b_\eta - b_\rho a_\eta. \]

This gives us

\[
v_\rho = \left( \frac{1}{a_\rho b_\eta - b_\rho a_\eta} \right) \bar{v}_\rho, \quad \text{and} \]

\[
v_\eta = \left( \frac{1}{a_\rho b_\eta - b_\rho a_\eta} \right) \bar{v}_\eta.
\]

Notice as well that

\[
\bar{v}_\rho \ast \bar{v}_\eta = (b_\eta, -a_\eta) \ast (-b_\rho, a_\rho)
\]

\[
(\bar{v}_\eta \ast \bar{v}_\eta) = (b_\rho a_\eta + a_\rho b_\eta)
\]

\[
= \frac{-(a_\rho a_\eta + b_\rho b_\eta)}{a_\rho^2 + b_\rho^2}
\]

\[
= \frac{-u_\eta \ast u_\rho}{u_\rho \ast u_\rho}.
\]

Putting this all together, we get

\[
\frac{v_\rho \ast v_\eta}{v_\eta \ast v_\eta} = \left( \frac{1}{a_\rho b_\eta - b_\rho a_\eta} \right)^2 \frac{(\bar{v}_\rho \ast \bar{v}_\eta)}{(\bar{v}_\eta \ast \bar{v}_\eta)} = \frac{v_\rho \ast v_\eta}{v_\eta \ast v_\eta} = \frac{-u_\eta \ast u_\rho}{u_\rho \ast u_\rho}.
\]

This way, we can rewrite (2.1) only in terms of ray generators:
\[
\text{Vol}_{\sigma}(P_{\sigma^*, \sigma^*}(z^\rho)) = z_\rho \frac{v_\rho \ast v_\eta}{v_\eta \ast v_\eta} + z_\eta = z_\rho - z_\eta \frac{u_\eta \ast u_\rho}{u_\rho \ast u_\rho}.
\] (2.2)

This represents the length of a face \( F^\rho(P_{\sigma^*}(z)) \) of \( P_{\sigma^*}(z) \). Recall that the face of \( C_{\Sigma^*}(z) \) associated to \( \rho \) is

\[
F^\rho(C_{\sigma^*}(z)) = \bigcup_{\sigma \in \mathcal{N}_\rho \Sigma(2)} F^\rho(P_{\sigma^*}(z))
\]

Then the volume of \( F^\rho(C_{\sigma^*}(z)) \) (weighted by \( \omega \)) is the sum of the formula in (2.2) over each maximal cone \( \sigma \) containing \( \rho \) as a face.

\[
\text{Vol}(F^\rho(C_{\Sigma^*}(z)); \omega) = \sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) \text{Vol}_{\sigma^*}(P_{\sigma^*, \sigma^*}(z^\rho)) = \sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) \left( z_{\sigma \setminus \rho} - z_\rho \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} \right).
\]

We officially have the volume of the face of \( C_{\Sigma^*}(z) \) associated to \( \rho \) only in terms of the generating rays. However, we can do some further clean-up to make this formula a little nicer. Distributing the weight through the summand and commuting the sum, we have

\[
\sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) \left( z_{\sigma \setminus \rho} - z_\rho \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} \right) = \sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) z_{\sigma \setminus \rho} - \omega(\sigma) z_\rho \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho}
\]

\[
= \sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) z_{\sigma \setminus \rho} - z_\rho \sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho}.
\]

Since \( \Sigma \) is a tropical fan, the weighted sum of \( u_{\sigma \setminus \rho} \) lives in the linear span of \( \rho \). That is,

\[
\sum_{\substack{\sigma \in \Sigma(2) \\ \sigma \succ \rho}} \omega(\sigma) u_{\sigma \setminus \rho} = a_\rho u_\rho
\]
for some $a_\rho \in \mathbb{R}$. Note that vector projection $\text{pr}_\tau$ of vectors in $V$ onto $V_\tau$ is a linear transformation for any $\tau \in \Sigma$, so we have

$$
\sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} = \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} \left( u_\rho \ast u_\rho \right)
$$

$$
= \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) \text{pr}_\rho(u_{\sigma \setminus \rho})
$$

$$
= \text{pr}_\rho \left( \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) u_{\sigma \setminus \rho} \right)
$$

$$
= \text{pr}_\rho(a_\rho u_\rho) = a_\rho u_\rho.
$$

Therefore

$$
\sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} = a_\rho.
$$

This equivalence allows us to rewrite our weighted volume formula one more time:

$$
\sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) z_{\sigma \setminus \rho} - z_\rho \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) \frac{u_{\sigma \setminus \rho} \ast u_\rho}{u_\rho \ast u_\rho} = \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) z_{\sigma \setminus \rho} - a_\rho z_\rho.
$$

**Lemma 2.1.** Let $\Sigma$ be a simplicial, 2-dimensional tropical fan in $V$, $\ast$ an inner product on $V$, $z \in \overline{\text{Cub}}(\Sigma, \ast)$, and $C_{\Sigma, *}(z)$ the normal complex with respect to $\ast$ and $z$. Then the weighted volume of the facet $F^\rho(C_{\Sigma, *}(z))$ is

$$
\text{Vol}(F^\rho(C_{\Sigma, *}(z)); \omega) = \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma) z_{\sigma \setminus \rho} - a_\rho z_\rho.
$$
2.2 Minkowski Weights for Incomplete 2-Fans

Now that we’ve carved up a clean formula for the volume of a facet of a 2-dimensional normal complex, we have enough machinery to prove the 2-dimensional case of the titular theorem.

**Theorem 2.1.** Let \((\Sigma, \omega)\) be a simplicial, 2-dimensional tropical fan in \(V\), \(\ast\) an inner product on \(V\), \(z \in \overline{\text{Cub}}(\Sigma, \ast)\), and \(C_{\Sigma, \ast}(z)\) the normal complex of \(\Sigma\) with respect to \(\ast\) and \(z\). Define the weight function

\[
\omega_{\text{Vol}} : \Sigma(1) \longrightarrow \mathbb{R}
\]

\[
\rho \longmapsto \text{Vol}(F^\rho(C_{\Sigma, \ast}(z)) ; \omega).
\]

Then \(\omega_{\text{Vol}}\) is a Minkowski 1-weight on \(\Sigma\).

**Proof.** We show that

\[
\sum_{\rho \in \Sigma(1)} \omega_{\text{Vol}}(\rho)u_\rho = 0.
\]

Since \(\Sigma\) is a 2-fan, every \(\sigma \in \Sigma(2)\) can be written as the Minkowski sum of two rays: \(\sigma = \rho + \eta\), where \(\rho, \eta \in \Sigma(1)\). Further, from Lemma 2.1, we know that for any \(\rho \in \Sigma(1)\),

\[
\text{Vol}(F^\rho(C_{\Sigma, \ast}(z)) ; \omega) = \sum_{\substack{\sigma \in \Sigma(2) \\
\sigma \succ \rho}} \omega(\sigma)z_{\sigma \setminus \rho} - a_\rho z_\rho.
\]

Then letting \(\sigma = \rho + \eta\), we can rewrite the sum in terms of rays:

\[
\text{Vol}(F^\rho(C_{\Sigma, \ast}(z)) ; \omega) = \sum_{\substack{\eta \in \Sigma(1) \\
\rho + \eta \in \Sigma(2)}} \omega(\sigma)z_\eta - a_\rho z_\rho.
\]
Similarly, we can rewrite the balancing condition around $\rho$ in terms of $\rho$ and $\eta$:

$$\sum_{\sigma \in \Sigma(2) \setminus \rho} \omega(\sigma) u_{\sigma \setminus \rho} = \sum_{\eta \in \Sigma(1)} \omega(\sigma) u_{\eta} = a_{\rho} u_{\rho}.$$  

Using this, we verify the balancing condition of $\omega_{\text{Vol}}$.

$$\sum_{\rho \in \Sigma(1)} \omega_{\text{Vol}}(\rho) u_{\rho} = \sum_{\rho \in \Sigma(1)} \left( \sum_{\eta \in \Sigma(1)} \omega(\sigma) z_{\eta} - a_{\rho} z_{\rho} \right) u_{\rho}$$

$$= \sum_{\rho \in \Sigma(1)} \left( \sum_{\eta \in \Sigma(1)} \omega(\sigma) z_{\eta} u_{\rho} \right) - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}$$

$$= \sum_{\rho, \eta \in \Sigma(1)} \omega(\sigma) z_{\eta} u_{\rho} - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}.$$  

In the first summand of the last expression, we commute the terms of the sum.

$$\sum_{\rho, \eta \in \Sigma(1)} \omega(\sigma) z_{\eta} u_{\rho} - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}$$

$$= \sum_{\rho, \eta \in \Sigma(1)} \omega(\sigma) z_{\rho} u_{\eta} - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}$$

$$= \sum_{\rho \in \Sigma(1)} z_{\rho} \left( \sum_{\eta \in \Sigma(1)} \omega(\sigma) u_{\eta} \right) - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}$$

$$= \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho} - \sum_{\rho \in \Sigma(1)} a_{\rho} z_{\rho} u_{\rho}$$

$$= 0.$$
as desired.

**Example 2.2.** Consider our familiar balanced fan \( \Sigma \subseteq \mathbb{R}^3 \) as in (1.6). Let \( * \) be the dot product on \( \mathbb{R}^3 \), and let \( z = (2, 2, 2, 3, 3, 3) \). Shown below is the normal complex \( C_{\Sigma,*}(z) \) with its facets labelled.

![Diagram of normal complex](image_url)

The ray generators of \( \Sigma \) are as follows:

\[
\begin{align*}
    u_{\rho_1} &= (1, 0, 0) & u_{\rho_0} &= (-1, -1, -1) \\
    u_{\rho_2} &= (0, 1, 0) & u_{\rho_{01}} &= (0, -1, -1) \\
    u_{\rho_3} &= (0, 0, 1) & u_{\rho_{02}} &= (-1, 0, -1) \\
    u_{\rho_{123}} &= (1, 1, 1) & u_{\rho_{03}} &= (-1, -1, 0).
\end{align*}
\]

The volumes of the facets can be computed via the formula in lemma (2.1), and are as follows:

\[
\begin{align*}
    \text{Vol}(F_1) &= 4 & \text{Vol}(F_0) &= 3 \\
    \text{Vol}(F_2) &= 4 & \text{Vol}(F_{01}) &= 2
\end{align*}
\]
Vol($F_3$) = 4    Vol($F_{02}$) = 2
Vol($F_{123}$) = 3    Vol($F_{03}$) = 2.

We verify that these face volumes give Minkowski 1-weights on the rays of $\Sigma$:

$$\sum_{\rho \in \Sigma(1)} \omega_{\text{Vol}(\rho)} u_\rho$$

$$= \text{Vol} (F_1) u_{\rho_1} + \text{Vol} (F_2) u_{\rho_2} + \text{Vol} (F_3) u_{\rho_3} + \text{Vol} (F_{123}) u_{\rho_{123}}$$

$$+ \text{Vol} (F_0) u_{\rho_0} + \text{Vol} (F_{01}) u_{\rho_{01}} + \text{Vol} (F_{02}) u_{\rho_{02}} + \text{Vol} (F_{03}) u_{\rho_{03}}$$

$$= 4(1, 0, 0) + 4(0, 1, 0) + 4(0, 0, 1) + 3(1, 1, 1) + 3(-1, -1, -1)$$

$$+ 2(0, -1, -1) + 2(-1, 0, -1) + 2(-1, -1, 0)$$

$$= (4, 0, 0) + (0, 4, 0) + (0, 0, 4) + (3, 3, 3) + (-3, -3, -3)$$

$$+ (0, -2, -2) + (-2, 0, -2) + (-2, -2, 0)$$

$$= (0, 0, 0).$$

2.3 Related and Future Work

In this section, we relate Minkowski’s theorem for incomplete tropical fans to existing concepts in algebraic geometry, namely tropical intersection theory. Next, we use Minkowski’s theorem for incomplete tropical fans to work toward one direction of an isomorphism between an algebra of normal complexes and a Chow ring, both associated to the same fixed fan.
Related Work

Here, we discuss similar and much more general structure assigned to tropical fans. Given a simplicial tropical $d$-fan $(\Sigma, \omega)$, define a rational function on $\Sigma(n)$ to be a continuous piecewise linear function $\varphi : \Sigma(n) \to \mathbb{R}$ on the cones $\tau \in \Sigma(n)$. For each cone $\sigma \in \Sigma$, denote $\varphi_\sigma$ to be the 'pieces' of the piecewise function. Given a rational function $\varphi$ on $\Sigma$, define a weight function

$$
\omega_\varphi : \Sigma(n-1) \longrightarrow \mathbb{R}
$$

$$
\tau \longmapsto \sum_{\sigma \in \Sigma(n) \atop \tau \prec \sigma} \varphi_\sigma(\omega(\sigma)u_{\sigma \setminus \tau}) - \varphi_\tau \left( \sum_{\sigma \in \Sigma(n) \atop \tau \prec \sigma} \omega(\sigma)u_{\sigma \setminus \tau} \right). \quad (*)
$$

The work of Lars Allermann and Johannes Rau in [1] shows that this weight is a Minkowski $(n-1)$-weight.

How does any of this relate to face volume? As it turns out, the formula we found for the volume of facets of a 2-dimensional normal complex is of the form $(*)$, which we show here.

Let $(\Sigma, \omega)$ be a tropical 2-fan. If we define a rational function $\varphi$ on $(\Sigma, \omega)$ by $\varphi(u_\rho) = z_\rho$ for all $\rho \in \Sigma(1)$, then we have

$$
\sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma)z_{\sigma \setminus \rho} - a_\rho z_\rho = \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma)\varphi_\sigma(u_{\sigma \setminus \rho}) - a_\rho \varphi_\rho(u_\rho)
$$

$$
= \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma)\varphi_\sigma(u_{\sigma \setminus \rho}) - \varphi_\rho(a_\rho u_\rho)
$$

$$
= \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma)\varphi_\sigma(u_{\sigma \setminus \rho}) - \varphi_\rho \left( \sum_{\sigma \in \Sigma(2) \atop \sigma \succ \rho} \omega(\sigma)u_{\sigma \setminus \rho} \right).
$$
One method of proving the conjecture (2.1) is to show that face volume of any dimension of normal complex falls into this form, since these are known to be Minkowski weights.

Normal Complex Algebras

In 1989, McMullen [5] defined an algebra of indicator functions of polytopes, which we will now describe. Denote by \([P]\) the indicator function on the polytope \(P \subseteq \mathbb{R}^d\) and \(\mathcal{P}_d\) the set of \(d\)-dimensional polytopes \(P \subseteq \mathbb{R}^d\). Define a commutative ring \(S\mathcal{P}_d\) generated by indicator functions on elements of \(\mathcal{P}_d\), where addition is formal, and the product \([P][Q] = [P + Q]\) is given by Minkowski addition. Furthermore, define an ideal

\[
\mathcal{T} := \mathbb{Z}\{[P + t] - [P] : P \in \mathcal{P}_d, t \in \mathbb{R}\} \subseteq S\mathcal{P}_d.
\]

Finally, define the polytope algebra:

\[
\Pi^d := \frac{S\mathcal{P}_d}{\mathcal{T}}.
\]

Elements of this algebra famously have an unusual nilpotent property. Let \(1 \in \Pi^d\) denote the equivalence class of the indicator function on a point, and let \([[P]] \in \Pi^d\) denote the equivalence class of the indicator function of a \(d\)-dimensional polytope \(P \subseteq \mathbb{R}^d\). Then

\[
([[P]] - 1)^{d+1} = 0 \in \Pi^d.
\]

Fixing a simple polytope \(P\), let \(\Pi^d(P)\) denote the subalgebra of \(\Pi^d\) generated by \([Q] \in \Pi^d\) such that \(P = \lambda Q + R, \lambda \in \mathbb{Q}\) and \(R \subseteq \mathbb{R}^d\) some polytope. Using that nilpotency, in 1993,
McMullen [4] defined an isomorphism between $\Pi^d(P)$ and the Chow ring $A^*(\Sigma_P)$ associated to the normal fan $\Sigma_P$ of $P$.

The more general conjecture (2.1) is posed with the intent to eventually define an isomorphism between an algebra comprised of classes of indicator functions on normal complexes associated to a fixed fan $\Sigma$ and the Chow ring associated to that same fan $\Sigma$. If the conjecture holds true for all dimensions, then an injection between the proposed algebra and the Chow ring will follow. A map from the Chow ring to the described algebra requires showing that the nilpotent property also holds for classes of indicator functions on normal complexes.
Bibliography


