Experimental and Computational Analysis
of Periodic Laminated Fiber-Reinforced Composite Beams

A thesis submitted in partial fulfilment of the requirements
For the degree of Master of Science in Mechanical Engineering

By
Julian Rodriguez

May 2020
The thesis of Julian Rodriguez is approved:

Dr. Tzong-Ying Hao

Date

Dr. Jaime Booth

Date

Dr. Peter Bishay, Chair

Date

California State University, Northridge
Acknowledgments

I would like to thank a few of people, that without their support and guidance I would have not be in the position I am in today. First and foremost, I would like to give a huge acknowledgment to my mentor Dr. Peter Bishay. The opportunity to research under his supervision has helped me leaps and bounds past what I thought I was capable of. His guidance in difficult times through the process, continuing to push me forward during my thesis work, even when I believed to be stuck, was an amazing aid.

I would like to thank Doris Chaney. She was one of the first to really see my potential in engineering. In my first year at CSUN, she gave me the opportunity to tutor others, allowing me to gain more knowledge and pass it on to my fellow peers. She was also the first to really introduce me to getting my Master’s degree, and without that, I do not know where I would have been today.

I would like to thank Dr. Tzong-Ying Hao and Dr. Jamie Booth, the thesis committee members, for reading the manuscript and giving ample amounts of guidance. Finally, I would like to acknowledge the RSCA grant Dr. Bishay received for this research and the financial support given by CSUN during my Master’s program.
Dedications

I would like to dedicate my thesis to my parents. They have been a huge support at home and at school. Without them, I wouldn’t even be here. Their push for me to stay in school to improve my knowledge, skills and abilities has been a huge support. They allowed me to work towards what I am happy doing. There is no one better I can ask for in my life.
# Table of Contents

Signature Page iii
Acknowledgments iv
Dedications v
Table of Contents vi
List of Figures ix
List of Tables xii
List of Symbols and Acronyms xiii
Abstract xiv

## Chapter 1 - Introduction

1.1 - Literature Review on Composite Structures/ Beams 1
1.2 - Literature Review on Periodic Structures 3
1.3 - Periodic Structures 7
1.4 - Composites 9
   1.4.1 - Layup 10
   1.4.2 - Curing 11
1.5 - Introduction to Experimental Modal Analysis 13
   1.5.1 - Data Acquisition Software 15
   1.5.2 - Channel Controls Tab 17
   1.5.3 - Generator Tab 18
   1.5.4 - Coherence 18
1.6 - Composite Theory 19
1.7 - Weak formulation for a Beam Finite Element 23
   1.7.1 - Shape Functions: Beam Bending 23
   1.7.2 - Beam Stiffness 25
   1.7.3 - Beam Mass 26
   1.7.4 - Shape Functions: Shaft 26
   1.7.5 - Shaft Stiffness 28
1.7.6 - Shaft Mass

1.8 - Dynamic Stiffness and Transfer Matrix

Chapter 2 - SOLIDWORKS Modal Analysis

2.1 - Model Description
2.2 - Convergence
2.3 - Verification
   2.3.1 - Isotropic Beam
   2.3.2 - Composite Beam
2.4 - Sweep Analysis
   2.4.1 - Effect of Periodic Ply Angle
2.5 - Clamp Vibration Analysis
   2.5.1 - Design of the Clamp
   2.5.2 - Clamp Modal Analysis
2.6 - Discussion

Chapter 3 - In-House FE Vibration Analysis

3.1 - Model Description
3.2 - Model Verification
3.3 - Sensitivity Analysis
   3.3.1 - Effect of PSR on Stopbands
   3.3.2 - Effect of PPA on Stopbands
   3.3.3 - Effect of NPL on Stopbands
   3.3.4 - Effect of NC on Stopbands
3.4 - One Dimensional Rectangular Cross-Sectional Shafts
   3.4.1 - Verification
3.5 - Discussion

Chapter 4 - Experimental Work

4.1 - Layup
4.2 - Material Processing 56
4.3 - Setup 57
4.4 - Vibration Analysis Verification 61
  4.4.1 - Isotropic Beam Testing 61
4.5 - Results 61
4.6 - Discussion 63

Chapter 5 - Conclusion 64
References 66
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>First depiction of stop bands and propagation factor in literature [35]</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Wave propagation through medium changes</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>Composite wind turbine blades. (picture animalworld.com)</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>Material and sequence of flat plate composite layup</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>Overhead view of composite layup order of materials</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>Left: ASC Autoclave. Right: Espec reach-in thermal chamber, Platinum Series</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>Two step curing cycle [55].</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>Vibration table [56]</td>
<td>13</td>
</tr>
<tr>
<td>9</td>
<td>Miniature shear accelerometer [57]</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>Test hammer (left) from PCB Piezoelectronics [58] and Smart Shaker [59]</td>
<td>14</td>
</tr>
<tr>
<td>11</td>
<td>Data Physics Ace Quattro signal processor [60]</td>
<td>15</td>
</tr>
<tr>
<td>12</td>
<td>SignalCalc new test screen</td>
<td>15</td>
</tr>
<tr>
<td>13</td>
<td>Initial Setup screen</td>
<td>16</td>
</tr>
<tr>
<td>14</td>
<td>Measurement Panel</td>
<td>17</td>
</tr>
<tr>
<td>15</td>
<td>Channel Controls Group</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>Generator Tab of SignalCalc Software</td>
<td>18</td>
</tr>
<tr>
<td>17</td>
<td>Example of a coherence graph</td>
<td>18</td>
</tr>
<tr>
<td>18</td>
<td>Graph to show representation of attenuation factor plotted over a plain and periodic beam</td>
<td>32</td>
</tr>
<tr>
<td>19</td>
<td>SOLIDWORKS mode made with surface patterns</td>
<td>33</td>
</tr>
<tr>
<td>20</td>
<td>Convergence plot of first mode frequency</td>
<td>33</td>
</tr>
<tr>
<td>21</td>
<td>First and Second Mode frequency vs periodic ply angle</td>
<td>35</td>
</tr>
<tr>
<td>22</td>
<td>Third and Fourth mode frequency vs periodic ply angle</td>
<td>36</td>
</tr>
<tr>
<td>23</td>
<td>SOLIDWORKS model of the clamp base</td>
<td>37</td>
</tr>
<tr>
<td>24</td>
<td>Second mode of the clamp</td>
<td>37</td>
</tr>
<tr>
<td>25</td>
<td>Dimensions and nomenclature for periodic beam with composites</td>
<td>39</td>
</tr>
<tr>
<td>26</td>
<td>Effect of PSR on first stop band.</td>
<td>41</td>
</tr>
<tr>
<td>27</td>
<td>Effect of PSR on second stop band.</td>
<td>42</td>
</tr>
<tr>
<td>28</td>
<td>First stopband, sweeping PSR while changing NC</td>
<td>42</td>
</tr>
<tr>
<td>29</td>
<td>Second stopband, sweeping PSR while changing NC</td>
<td>42</td>
</tr>
</tbody>
</table>
Figure 30: First stopband, sweeping PSR while changing NPL
Figure 31: Second stopband, sweeping PSR while changing NPL
Figure 32: First stopband, sweeping PSR while changing PPA
Figure 33: Second stopband, sweeping PSR while changing PPA
Figure 34: First Stop band with sweeping ply angle.
Figure 35: First and second stop bands with sweeping periodic ply angle.
Figure 36: First stopband, sweeping PPA while changing NC
Figure 37: Second stopband, sweeping PPA while changing NC
Figure 38: First stopband, sweeping PPA while changing NPL
Figure 39: Second stopband, sweeping PPA while changing NPL
Figure 40: First stopband, sweeping PPA while changing PSR
Figure 41: Second stopband, sweeping PPA while changing PSR
Figure 42: First Stop band with sweeping number of layers.
Figure 43: First two stop bands with sweeping number of layers.
Figure 44: First stopband, sweeping NPL while changing NC
Figure 45: Second stopband, sweeping NPL while changing NC
Figure 46: First stopband, sweeping NPL while changing PPA
Figure 47: Second stopband, sweeping NPL while changing PPA
Figure 48: First stopband, sweeping NPL while changing PSR
Figure 49: Second stopband, sweeping NPL while changing PSR
Figure 50: First stop band with sweeping number of cells (NC)
Figure 51: First two stop bands with sweeping number of cells (NC)
Figure 52: First stopband, sweeping NC while changing NPL
Figure 53: Second stopband, sweeping NC while changing NPL
Figure 54: First stopband, sweeping NC while changing PPA
Figure 55: First stopband, sweeping NC while changing PPA
Figure 56: First stopband, sweeping NC while changing PSR
Figure 57: Second stopband, sweeping NC while changing PSR
Figure 58: Alpha vs height-to-width ratio of rectangular beam
Figure 59: Layup of composite structure as a single unit
Figure 60: Layup of composite structure without lower periodic patches
Figure 61: Application of cyanoacrylate to adhere composite periodic section to Aluminum beam 56

Figure 62: Hole drilled for force sensor mounting 57

Figure 63: Completed periodic beam 57

Figure 64: Completed shaker setup 58

Figure 65: Needed Equipment, minus cabling and beam, for shaker testing 58

Figure 66: Periodic beam attached to clamp 59

Figure 67: Showing location of accelerometer and tape to hold down wire 60

Figure 68: Aligning Stringer to force sensor 60

Figure 69: Computational frequency analysis on periodic and non-periodic composite beams 62

Figure 70: Experimental frequency analysis on periodic and non-periodic composite beams, highlighting area of stop bands 62
List of Tables

Table 1: Comparison of 6061 Aluminum cantilever beam to analytical solution 34
Table 2: Comparison of ANSI 1020 Steel cantilever beam to analytical solution 34
Table 3: Comparison of natural frequencies of a simply supported 0-degree composite beam 35
Table 4: Modal Frequencies and Shapes of clamp 37
Table 5: Analytical vs numerical modal frequencies for cantilever carbon fiber composite beam. 40
Table 6: Cantilever E-glass/epoxy, length 0.4m, width 0.05m, NPL 2, NML 2, PSR 0.4, NC 5, PPA 0 40
Table 7: Cantilever E-glass/epoxy, length 0.4m, width 0.05m, NPL 2, NML 2, PSR 0.4, NC 5, PPA 30 41
Table 8: First twist frequency of ANSI 1020 steel beam with rectangular CS and sweeping height 53
Table 9: First twist frequency of 6061 Al. Alloy beam with rectangular CS and sweeping height 53
Table 10: First twist frequency of E-glass epoxy beam with rectangular CS and sweeping height 53
Table 11: List of equipment for modal shaker testing 59
Table 12: Comparison between numerical and experimental bending frequencies 17.75" × 1.71" × 0.09" 6061 Aluminum beam 61
List of Symbols and Acronyms

MLM – Main Layers Material
MLT – Main Layers Thickness
PLM – Periodic Layers Material
PLT – Periodic Layers Thickness
NLM – Number of Layers in Main Beam
NPL – Number of Layers in Periodic Patches
PLA – Periodic Layers Angles
PSR – Periodic Segment Ratio
PPA – Periodic Ply Angles
NC – Number of Cells
Π – Total potential of the system
U – Strain energy of system
T – Kinetic energy of the system
W – Potential energy of load system
FEA – Finite element analysis
PDS – Product data sheet
DAQ – Data acquisition system
FRP – Frequency response plot
\( E_x \) – Elastic modulus in X
\( E_y \) – Elastic Modulus in Y
\( E_z \) – Elastic Modulus in Z
\( \nu_{xy} \) - Poisson ratio in XY
\( \nu_{yz} \) - Poisson ratio in YZ
\( \nu_{xz} \) - Poisson ratio in XZ
Abstract

Experimental and Computational Analysis
of Periodic Laminated Fiber-Reinforced Composite Beams

By
Julian Rodriguez
Master of Science in Mechanical Engineering

The dynamic behavior of solid structures is an important aspect that must be considered in
the design phase to ensure that the designed structure will have desired response under external
excitation. Periodic structures with varying geometries and materials have been examined under
analytical, numerical and experimental ways when looking into current literature. There has been
a confirmation of attenuation of wave propagation in periodic structures compared to non-periodic
ones. The inclusion of laminated fiber-reinforced composite materials to periodic structures are
yet to be heavily researched. Laminated fiber-reinforced composite materials, in general, give an
additional design characteristic, which is the stacking sequence of the plies. The ability to create
materials that work for a design instead of designing based on available materials makes them a
powerful solution to future problems. In this work, a commercial finite element analysis software
(FEA), SOLIDWORKS, was used to analyze the natural frequencies of periodic beams with
periodic laminated fiber-reinforced composite patched. Then with the use of composite lamination
theory and wave finite element (WFE) method, an in-house code was developed to plot bending
frequencies, and solve for “stop bands”. Along with the in-house code, an attempt to make a
simplified rectangular twist element using a stiffness factor relative to a circular shaft was done to
enable analyzing twisting modes of vibration. Finally, experimental studies were done to show the effectiveness of periodic wave guides.

Using a surface model, SOLIDWORKS allows composite materials to be defined and studied. The natural frequencies agreed with isotropic cantilever beam bending frequencies and composite beam analytical theory. When sweeping angle of the periodic ply, and viewing the change in frequencies, it was shown, for a laminated fiber-reinforced periodic cantilever beam, that the angle of the plies affects the natural frequencies less as number of plies increase.

A sensitivity analysis was done investigating the effects of periodic beam parameters on stopbands, using the in-house MATLAB code and a laminated fiber-reinforced composite material. Using formulas for effective stiffnesses for symmetric laminated composites, a beam element was defined. The element was seen to have agreement with analytical solutions within 2.75% for the first five bending modes. Number of plies (NP), periodic segment ratio (PSR), periodic ply angle (PPA), and number of cells (NC) were swept to show how each affects the location and width of the stopbands.

Experimental modal testing was conducted to verify the presence of the stop bands utilizing periodic fiber-reinforced composite patches. An aluminum beam was outfitted with carbon fiber-reinforced composite patches and compared to numerical results. With the use of an accelerometer, shaker and clamp system, a cantilever support was created and bending modes of vibration were plotted. A trend that resembles the computational results was seen, and a stop band was found in a similar frequency range as in the computational results.
1.1 - Literature Review on Composite Structures/Beams

Composites have been a focus of research due to their promise of new design choices and optimizations. The models for analyzing composite beams are generally put into three categories, classical lamination theory, first order shear deformation theory and higher-order shear deformation theory [1]. Reddy [2] derived a higher-order shear deformation theory for laminated composite plates like the first-order shear deformation theory in 1984. The theory presented, accounts for parabolic distribution of the transverse shear strains through the thickness of the plate. The theory is shown to give more accurate solution than the first-order theory. In 1990 Chandrashekhara, Krishnamurthy and Roy [3] derived and solved for the exact solutions for the free vibration of symmetrically laminated composite beams. This implements first order shear deformation and rotary inertia. This was then presented for a variety of boundary conditions and showed the effect of shear deformation and material anisotropy. Later, another exact solution was developed in 1997 by Banerjee and Williams [4] who developed an exact dynamic stiffness matrix (DSM) method for free vibration of simple structures comprised of composite beams. The model used included the bending-torsion coupling of the beams due to the material coupling being composite seen in [5]. Other exact solutions are available for solving specific orientations of composite plies. For example, Khdeir and Reddy [6] solved for the bending modes of cross-ply laminated composite beams in 1997. They went through the first through third order shear deformation theories of analysis for solving the governing equations. Effects of shear deformation is seen to be important between theories when length-to-height ratio decreases. Newer theories have been created to investigate wave propagation in laminated composites. Nanda, Kapuria and Gopalakrishnan [7] devised a spectral finite element model based on an efficient layer wise theory to analyze wave propagation through composite and sandwich beams, shortened to ZIGT. This theory differs from first order shear deformation or the global third order [2], as transverse shear is continuous at the layer interfaces.

Later, finite element methods were employed to solve for dynamic motion of laminated beams. In 2002, Chakraborty, Mahapatra and Gopalakrishnan [8] analyzed free vibration and wave propagation in asymmetric composite beams. This element was shown to be free of shear locking and allow for asymmetric layer choices called the refined first order deformation theory element, “RFSDTE”. The element was tested under various boundary conditions and compared to the exact
solutions from [3]. This was then used as a basis for many papers, as a reference, compared to numerical results. A year later, in 2003 Chakraborty, Gopalakrishnan and Reddy [9] defined a new beam finite element model for composites. This element was specified for functionally graded materials, “FGM”. This research was to study the thermoelastic behavior of FGM’s as a beam element. Functionally graded materials are defined in the paper as materials or structures in which the material properties vary with location. They are used heavily in thermal applications, which metallic and ceramics comprise the material. As time progressed, the use of dynamic stiffness methods was introduced to composites. Banerjee, Su and Jayatunga [10] investigated the free vibration characteristics of composite beam and their applications to aircraft wings using the dynamic stiffness method. In the same year, Jun, Hongzing and Rongying [11] utilized first order shear deformation theory to create a DSM for investigating free vibration of generally laminated composite beams. The dynamic stiffness in this formulation becomes quite lengthy and symbolic manipulation using Mathematica were used to solve for the matrix.

More recently, Osman and Suleiman [12] used the finite element method to analyze free vibrations of laminated composite beams in 2017. With the use of first order deformation theory, they solved the differential equations of motion for composite beams with “regular” cross-sections. This developed a 9\times9 stiffness and mass matrix for analysis. In 2018, Babuska, Wiebe and Motley [13] went through the creation of an accurate bend-twist coupled composite laminated beam. They stated that, although numerical and analytical studies have been done on the dynamics of similar beams, they can be “cumbersome to implement quickly and efficiently.” Their element has two nodes with 6 degrees of freedom each. This element used bend-twist coupling rigidity, $K$, elastic modulus in the $x$ direction and St. Venant’s torsional rigidity. This all leads to 20 shape functions. Being a finite element method, it requires increasing elements to get more accurate results.

Functionally grade materials, viscoelastic sandwich composites and delaminated laminated composites have also been researched in the same category and gave a great insight to methods to analyze composite structures. In 2012 Golub, Fomenko et al. [14] studied time-harmonic plane elastic SH-waves in periodically laminated composites with functionally graded layers. This EFG method was compared well with a laminated rod obtained using spectral finite element method. It was also said that the method could be applied to laminated structures with defects and damages. Viscoelastic sandwich beams are made like composites and are often lumped in the studies. Papers that deal with cylindrical shells or laminates have been studied since nearly 1965 through today
With delamination being an issue with composites, Erdelyi and Hashemi [20] created a dynamic stiffness element to examine delamination in composite beams. This element utilized the Euler-Bernoulli theory and two layered composite beams. Compared to FEM models and other numerical methods, the DSM was found to be in good agreement. Delamination results in a coupling between axial and transverse motion in these segments.

1.2 - Literature Review on Periodic Structures

Wave propagation in beams of different materials or geometries is a very important research topic. The formulations to solve for the behavior of the waves is quite extensive. The finite element methods to solve the free vibration of beams is done using, most commonly, Euler-Bernoulli theory or Timoshenko theory. Using these theories, stiffness matrices, dynamic stiffness matrices or spectral elements can be produced. In 1985, Banerjee and Williams [21] obtained expressions for the exact dynamic stiffness for a beam with a tapered section. The tapered sections have a specific dimension equation for $GJ$, $I$ and across-sectional area $A$. To solve the frequency equations, Bessel functions were used. This gave a 7-figure accuracy compared to an approximated tapered beam. Then, in 1989, as an application for the dynamic stiffness matrix, Langley [22] investigated free and forced vibrations of aircraft panels. These panels are usually stiffened along edges or transversely. The method is used to find the response of panel rows. Also, a dynamic stiffness for a stiffener is derived allowing for the effect of prestress. Banerjee el al. [23] produced a few other papers including, in 1994, the analysis of Timoshenko beam elements coupled in bending and torsion while being axially loaded. This could be used for applications such as turbine blades. In 1997, Banerjee [24] defined the dynamic stiffness matrix of a structural element. The use of the dynamic stiffness matrix is a way to solve free vibration problems of structures. The dynamic stiffness eliminated the separation of mass and stiffness matrices. The benefit of this matrix is that it is independent of the number of elements used in the analysis, which reduced computation time. And then as current as 2014, Gao [25] created a new Timoshenko beam model that incorporates microstructure and surface energy effects. By looking into the microstructure, there are macroscopic phenomena that are explained, while surface energy allows for surface elastic constants and residual stresses to be accounted for. The use of the transfer matrix was a big discovery and its use with wave propagation allowed a medley of studies to be conducted. The transfer matrix has been even used by Wang, et al. [26] for discussing wave propagation in plate covered by periodic layers. They created a theoretical model to describe this wave propagation,
using the transfer matrix method. The results showed that with a few material and geometric properties the width and location of the bandgaps can be varied.

When looking into periodic structures and more specifically beams, we must look at the history of the concept. As early as 1887, the first mention of continuous periodic structures was noted by Lord Rayleigh [27]. Rayleigh utilized Hills method to solve a second order governing wave equation with damping and restitution coefficients. He then went on to show a solution that describes reflected waves based on periodic geometry. Further in the future, in 1964, Heckl [28] investigated vibrations in commonly used cross-section beams and grillages. These are types of two-dimensional periodic systems, while he did also look into bending waves in beams with periodic discontinuities. After this, in 1969, Gupta [29] explored beams and plates with periodic supports. This work looked at the extreme ends of the beams and showed that only certain discrete values of the propagation constant satisfy the conditions at the end. He explained the meaning of the propagation factor and states that the propagation factor, $\mu$, is complex in general. The real part defines the rate of decay of the amplitude from one support to another and the imaginary part denotes the phase difference between the quantities at two successive supports. In 1973, Mead [30] showed a general theory of wave propagation in one-dimensional periodic systems with multiple coupling between adjacent elements. It is explained that the “propagation constant”, $\mu$, occurs in negative and positive pairs. It is derived as harmonic motion in one element and is equal to $e^{\mu}$ times the motion at the corresponding point in the next element and consists of a real and imaginary parts called “attenuation constant” and “phase constant”, respectively. The pairs define equal and opposite waves. Mead is a large contributor to the periodic structure research field. In 1986 he created a new method for analyzing wave propagation in periodic structures that are subjected to an array of harmonic forces or moments [31]. This was applied to Timoshenko beams and stiffened plates. In 1996, Langley [32] looked into the concept of the transfer matrix and showed that for a periodic structure, the energy flow velocity is equal to the group velocity. He, Rui and Zhang [33] applied the transfer matrix method to analyze a tree structure system in 2012. The tree systems consist of mass spring systems branching off an original spring support, looking like a biological tree.

Then in 1995, Mead [34] created a history of periodic structures and specifically, the contributions from Southampton. This history went through the first notes of periodic systems done by Sir Isaac Newton, though not called periodic structures yet, all the way to research done
at the time of publication in 1995. The first presented theory for stopbands was also shown, as in Figure 1.

![Figure 1: First depiction of stop bands and propagation factor in literature [35]](image)

Another review of periodic structures and the analysis of them was done by Mester and Benoroya [36] in 1995. They gave a review of different methods of analysis of one- and two-dimensional periodic structures. In 2006, Mace, Duhamel et al. [37] presented a method to predict wave numbers for a one-dimensional waveguide using a finite element model. This model utilized the conventionally found mass and stiffness matrices. This is different to the normal spectral element approach. The same year, Duhamel published another paper in which, Duhamel, Mace and Brennan [37] examined a post processing method for solving the forced response of a structure using the eigenvectors and global dynamic stiffness matrix. This approach is called the wave finite element or WFE. This is much more efficient as it focuses on one cell instead of the complete structure for analysis. A similar paper was published by Mace, Duhamel and Hinke [38] using the spectral element approach.

In more recent years, Lui and Hussein [39] examined the band gaps that appear in flexural beams and utilized Bloch’s theorem with Timoshenko beam theory for the motion of the beam. Width of the bandgaps were plotted for differing moduli, densities, areas, spring constants and
masses. It was stated that to enlarge the bandgap size it is not cost effective to increase the material properties past a certain value. Studies have not been kept to rectangular beams. Li, Chen et al. [40] examined the bandgap for torsional vibration of one-dimensional shafts in 2012. This method was a hybrid method combining the transfer-matrix and lumped-mass methods. The shafts have varying materials as the radius changes. It was shown to be in good agreement with finite element methods. Guo and Fang [41] presented an analysis method to study longitudinal waves in periodic multiphase rods. This method, called method of reverberation-ray matrix, or MRRM, agrees with the transfer matrix methods. Thus, this gave another form to be able to solve for longitudinal waves in periodic structures. The next year, Ding, Zhu et al. [42] explored ordered and randomly disordered jointed tunnel under moving loads. This can be very important for high-speed rail ways that produce noticeable vibrations. Design of periodic beams has also been conducted. Muzzaffaruddin and Bishay [43] investigated a design procedure for isotropic geometrically periodic beams in 2018. They utilized a “forward approach” by investigating how geometric properties affect the location of stop bands. A transfer matrix is used with a wave finite element approach to get stop bands. This approach showed zero, one or two solutions for a set of desired parameters.

The analysis of periodic beams has not been just in the form of isotropic beams. Composite beams have been constructed, some alternating materials (a form of composite) in each segment, while others using layering in each segment. Also, the use of smart materials has started to be introduced more recently. In 2016, Guo, Sheng and Wang [44] examined wave attenuation in periodic laminated beams. This utilized Hamilton’s principle, transfer matrix and then Bloch-Floquet boundary conditions. The beam was designed to be a single laminated beam, consisting of an equal number of layers in each segment. This is not fiber-reinforced but each layer can have its own moduli, density and shear coefficient. Analysis of a sandwich beam was considered and showed that almost half the waves below 2000 Hz could be suppressed for the specified beam examined. For lower frequencies, both the Timoshenko and Euler-Bernoulli models were shown to output similar results. Composite structures can also be made from alternating segments with limited layers. Along with Guo, in 2016, Song et al. [45] looked at the influence of mechanical and geometrical properties for periodic composite material beams wave propagation. These structures were affected by longitudinal waves with the composite structure comprised as each segment is a different material. Guo and Sheng [46] studied the bandgaps developed due to a
periodic bi-layers structure. The cell of this structure was comprised of 4 materials. The model used Timoshenko beam theory in both layers. No slip, small strains and displacements, and longitudinal vibration was considered. Differences were seen between this model and COMSOL models, possibly due to no bend-twist coupling. A parametric study was also done to better understand the bi-layer beam and used towards design. Recent research in composite periodic beams has been done by Zhang et al. [47] [48] in their paper, published in 2018. A new model was developed to find bandgaps for wave propagation in composite beam structures. This model incorporated microstructure, surface energy, and rotational inertial effects. Although it considered composites, the model utilized a microstructure approach. The model also used large matrices that would require more time to calculate and show frequency bandgaps. Tuning of beams, non-periodic and periodic, using shape memory alloys has also been investigated as describing the dynamic behavior or analyzing bandgaps [49] [50] [51].

Although extensive research has been done on experimenting the dynamic behavior of composite beams, such as [52], the addition of fiber-reinforced composite laminates to periodic structures and how those affect the stop bands has not been done. There has been an experimental study regarding isotropic periodic beams. Domadiya, Manconi et al. [53] studied the location of stop bands numerically and experimental. They investigated two theories to solve for stop bands. One is the use of a single cell wave finite element approach while the other is using vibration level difference, or LVD. This was done numerically and experimentally and both approaches were found to give matching results. Thus, with the introduction of fiber-reinforced composite periodic patches, experimentation to further ground the numerical results is necessary.

In summary, with the addition of composites to periodic beams analysis can get quite challenging. In this research, the analysis of periodic beams with the addition of composite materials will be done using a classical lamination theory method and finite elements. The analysis of not only stop bands but also modal behavior of the beams is conducted. Since limited research has been conducted on experimental analysis of periodic beam structures, the experimental work in this research can be used to further solidify the numerical analysis.

1.3 - Periodic Structures

When designing any type of system, there is a possibility for unwarranted vibrations that come into play. Vibrations are often caused by rotating shafts, earthquakes, or flutter on a wing of an aircraft. These vibrations will have large amplitudes when matched with resonant frequencies
of the structure. The phenomenon is called resonance. Resonance can then lead to failure. Active and passive vibration control have been implemented and are the most common tactics to attenuate vibrations.

Active vibration controls are implemented by dynamically reacting to incoming vibrations and impose an equal and opposite force to reduce vibrations. This is done using sensors and actuators. The actuators can be smart material actuators, commonly piezoelectric, magnetostrictive, electro-magnetic actuators, or electro-hydraulic actuators. Active vibrations have their own issues. Issues with active systems are their larger size, number of components, or the intricacy of the algorithm that must be made to combat the frequencies being applied.

Passive vibration damping is a form of reducing the vibrations forced onto the system of interest by adding additional passive components. This can be made with something as simple as a piece of rubber under the structure, a swinging mass in a building, or as intricate as a shock system in a car that utilizes fluid friction to dampen vibrations. These systems can be bulky, invasive, or insufficient to remove all vibrations on the system. Thus, creating a passive system that evades these issues is very desirable.

A lot of research in the field of periodic structures or wave guides (passive vibration filters) has been conducted in the past few decades due to their potential as passive vibration dampers. When a periodic structure vibrates, “band gaps” are formed in the frequency response plot to show how frequencies in specific regions of interest are attenuated in the vibrating system.

Mead [34] states that a periodic structure consists of several identical structural components which are joined together, end-to-end and/or side-to-side to form the main structure. Periodic structures can be placed in one of a few categories: one dimensional, two dimensional and three dimensional. One dimensional periodic structures include train tracks, beams, or linear spring mass systems. Two dimensional periodic structures can be visualized as solar panels, windows, or tiles on a floor. Finally, with the current increase in the use of 3D printing, the lattice structures they create internally for support can be classified as a three dimensional periodic structures. All these types of structures do not need to be limited to square structures, they could be shells, trusses, or round wheels. The perfect form of these structures is the structure of pure crystals.
Wave propagation in periodic structures is based on the physics principle that when waves travel across a discontinuity, whether that be a different medium or shape change, the incident wave continues in two different manners. One is continuing to be transmitted to the next medium, shown in Figure 2, and the other is reflected back.

![Figure 2: Wave propagation through medium changes](image)

This phenomenon is also seen when the waves propagate through a discontinuity in geometry. When the reflected waves encounter incident waves, they have the possibility of negating or reducing the amplitudes.

1.4 - Composites

Composites are defined when two or more materials are combined macroscopically to create a new material. This could be almost any combination of materials, for example, carbon fibers with a polymer, metal with metal or ceramic with metal. These types of materials have been recorded as far as structures have been made, such as buildings made with clay and hay mixed together to produce a more stable structure. Although there are downsides to every type of material, composites allow for much more design variation of materials and ability to tailor structures to the design goals.

Composite materials consist of two parts, the matrix and the reinforcement. Reinforcements can be fibers, either woven or unidirectional, spheres, flakes, or other shapes that work for the specific application. The characteristics of the composites is different from that of its constituents. For example, in the more commonly known composite material of epoxy matrix and glass fibers, this material uses the stiffness of glass and ductility of epoxy. This new material will be stiffer than glass but not as brittle, due to the presence of epoxy.

Utilization of composites in structures is not a new concept, from the concrete of the Pantheon to planes of today that are comprised upwards of fifty percent composite structure [54].
The composite structures of a plane wing or wind turbine blade, seen in Figure 3, are just two types of beam like designs that place the composites in a cantilever setup. Thus, static and dynamic studies of these types of structures are crucial to understanding their behavior and how to mitigate issues that arise from them. The manufacturing of composite structures used in this work will now be discussed.

![Composite wind turbine blades. (picture animalworld.com)](image)

**Figure 3: Composite wind turbine blades. (picture animalworld.com)**

1.4.1 - **Layup**

![Material and sequence of flat plate composite layup](image)

**Figure 4: Material and sequence of flat plate composite layup**

For a composite beam to be made, a general layup sequence is needed. This sequence includes the order of the materials needed and some methods to getting a successful layup. As seen in Figure 4, materials required are, from bottom to top, the mold coated with release agent, composite material, matte sheet (not required), release film, breather material and bagging material as a last layer. To hold down the bagging material, a bagging tape is used. The matte sheet can be used if a matte finish is wanted on the composite. Another view can be seen in Figure 5.
Figure 5 further shows the sequence and position of all the materials. In addition, a circular vacuum connector is shown. The vacuum connector is placed over the breather material and a hole is poked through the bagging to allow for a top piece to seal the connector. This is where a vacuum pump is connected to vacuum out all air from the layup. Each material is cut to a slightly larger size than the last. If possible, the vacuum connector can be placed in a location that allows for optimal composite material to be placed.

1.4.2 - Curing

Once the layup has been completed, the composite structure is cured. The curing cycle is a way for the epoxy to set. Heat and pressure are applied to the structure to reduce the number of voids possible from being created. The pressure is applied using a vacuum pump by removing air from the layup, allowing a pressure difference in the bag from outside, putting pressure on the composite. The best form of curing is done in an autoclave. An autoclave is a pressurized oven that allows not only heating but also increased pressure. A picture of an autoclave and non-pressurized oven can be seen in Figure 6.
These can be as small as a conventional oven all the way to the size of buildings needed to cure airplane components. Although autoclaves allow for the highest performance, they can be expensive since they require a lot of power, and the pressurization of the inside requires compressed air or Nitrogen gas. Nitrogen is quite expensive; thus, utilization of an autoclave is mainly efficient if large quantities or sizes of parts are to be made.

If an autoclave is not an option for the layup being executed, then using an oven is also an option. The oven in question does need to heat at a certain rate. A huge part of the curing is the heating rates, such as ramp up and ramp down rates. These rates are specified in product data sheets. The oven used for experiments in this work was an Espec Reach-in Thermal chamber, shown in Figure 6.

Once the oven was chosen, the curing process was decided upon. In a product data sheet of the composite in question, there should be a curing cycle with temperatures and pressures that can be used. These can be relayed in terms of tabulated data or graphs, of which can be seen in Figure 7.
1.5 - Introduction to Experimental Modal Analysis

To ground us to the physical world and reproduce trends or exact values, experimental work must be done. In the case of periodic analysis, vibrational testing must be conducted to excite frequencies developed in the structure and create frequency response plots. The experimental setup includes a clamp, data acquisition system (including software), accelerometers, and a shaker.

The clamping system is done in a way to recreate the boundary conditions like the theoretical models. These can be as simple as clamping the part to a table to be hit with a hammer or fixed to a vibrational table. Seen in Figure 8, is a vibrational table. A structure can be placed on the table and the whole table is excited, in turn, exciting the structure.

**Figure 7: Two step curing cycle [55].**

**Figure 8: Vibration table [56]**

Accelerometers are a way of measuring a system’s velocity change per unit time. Although acceleration cannot be measured directly, an accelerometer can measure the force input and convert this to an acceleration using $F = ma$. In this work, mini shear accelerometers were utilized
to sense motion in the beams. This accelerometer utilizes a ceramic piezoelectric material to be able to output a voltage response due to a force applied. This voltage can be converted to an acceleration.

![Miniature shear accelerometer](image)

*Figure 9: Miniature shear accelerometer [57]*

To excite the system, multiple types of tools could be used. The two main forms of input to a vibration system are shakers and impact hammers, both seen in Figure 10. Impact hammers are largely used equipment for exciting a structure. It is easy to use, and results are quick to obtain. Exact repeatable impacts are difficult and higher natural frequencies are less likely to be found since not enough energy is put in to excite them. Due to these issues, use of a shaker for excitation is performed as the source of vibrations. Shakers are attached via a stringer and a force sensor to the structure. Results can be narrowed to look for specific frequencies and these tests are extremely repeatable and give reliable results. A downside is that the setup and testing takes more time than the impact hammer.

![Test hammer and smart shaker](image)

*Figure 10: Test hammer (left) from PCB Piezoelectronics [58] and Smart Shaker [59]*
A data acquisition system was required in order to take the input of the accelerometer and produce a signal for the data processing software. The system can take in inputs from a multitude of sensors including force sensors and accelerometers. Pictured in Figure 11 is the Data Physics SignalCalc Ace Quattro which has 4 input channels and 2 sources. These systems can be as small or smaller than the one here or much larger. A larger example is the SignalCalc Savant which boasts a 40-1024 input channel ability.

![Data Physics Ace Quattro signal processor](Image)

*Figure 11: Data Physics Ace Quattro signal processor [60]*

1.5.1 - **Data Acquisition Software**

For software, a specific data acquisition program is used, called SignalCalc. This software is compatible with the Data Physics Signal Processor seen in Figure 11.

![SignalCalc new test screen](Image)

*Figure 12: SignalCalc new test screen*
When beginning the program, a test type must be selected. Since a force sensor and an accelerometer are used, *Transfer Function* is chosen to allow for a relative rating from the input force and acceleration to be received. This will allow scaled data that follow each other more inherently. After the test has been chosen the main initial setup page is shown, seen in Figure 13. Here is where most graphing can be seen, and other input data can be changed.

![Initial Setup screen](image)

*Figure 13: Initial Setup screen*

One of the important parts of the initial setup screen is the measurement panel, as seen in Figure 14. This panel allows one to change the frequency span, number of lines, and a multitude of other variables.
1.5.2 - **Channel Controls Tab**

The input tab is one in which all the sensors can be entered and activated. The main columns that are most relevant are, Coupling, mV/EU, and EU. These three define the main parts of the sensor used. The coupling tab is changing the way the sensor receives data. Both the force transducer and accelerometer are considered ICP, which place the amplifier as close to the sensor as possible. This allows the software to auto range the sensors and set their ranges. The mV/EU is the sensitivity of the specific sensor used. Sensitivity shows how much voltage is created in the sensor due to a specific unit input, such as a force. These sensitivities are provided in the sensors’ data sheets. Finally, the EU is the engineering unit, which is also provided by the manufacturer in the same location as the mV/EU.
1.5.3 - Generator Tab

![Generator Tab of SignalCalc Software](image)

In the generator tab, output data for the shaker is placed. With this software, multiple waveforms are available but for the experiments conducted, Swept Sine is chosen so that specific frequencies are brushed through. The level allows for the excitation to be lowered or raised based on how much force one would like to be applied. Freq and Freq2 are in place to put the lower and upper bounds on the frequencies to be excited. The Width and Width Unit are coupled, as the Width Unit has options to which the wave will be ramped up. The Lin Sec is chosen for a linear increase of frequency over the Width. The Width is the time allotted for the frequencies to go through.

1.5.4 - Coherence

In order to visualize whether the FRP that is being displayed is correct or without an abundance of noise, a coherence plot is created. This is a way to graph how the response is related to the input excitation. For a good coherence, the graph should read 1. This is not always the case in terms of noise being introduced, at resonances or anti-resonances. An example of a coherence graph can be seen in Figure 17.

![Example of a coherence graph](image)
1.6 - Composite Theory

With composites we must not view them as common isotropic materials, but when looking at the bending modes of a beam, we can assume the beam is isotropic with material properties equal to the effective material properties of the composite. A beam under bending is subjected to varying stress levels along the longitudinal axis of the beam. Thus, we can start with the most general material properties and derive the composite effective properties.

Starting with a single layer, or a lamina, with a completely anisotropic materials we can write the stress-strain relation in the lamina 1-2-3 directions as:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{13} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}
\]

(1)

When there are two orthogonal planes of material property symmetry, there will be a mutually orthogonal symmetric plane that exists. This reduces the stress-strain relations to:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{13} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}
\]

(2)

This is defined as an orthotropic material.

For a fiber-reinforced lamina we can assume plane-stress for a very thin structure such as unidirectional and woven laminated beams. For this, we impose that stress only is developed in the plane of the plate, thus the following assumptions can be made:

\[
\sigma_3 = 0; \tau_{23} = 0; \tau_{13} = 0 \quad \sigma_1 \neq 0; \sigma_2 \neq 0; \tau_{12} \neq 0
\]

This assumption reduces the strain-stress relations to:
\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
= \begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

There is an implied out-of-plane strain \( \varepsilon_3 = S_{13} \sigma_1 + S_{23} \sigma_2 \). This is now inverted to get the stress-strain relations:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

We can then transform these values into the global x-y coordinates, if the fiber orientation is skewed from the global x-y coordinates used. This new relation can be written as:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
= \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

where,

\[
\begin{align*}
\bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12} \left( \sin^4 \theta + \cos^4 \theta \right) \\
\bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66} \left( \sin^4 \theta + \cos^4 \theta \right)
\end{align*}
\]

These relations are utilized for a single layer of the composite. With this we can go into getting necessary information about a laminate, consisting of multiple layers.

Starting with some assumptions about the laminates, Kirchhoff-Love hypothesis for shells. This states that sections normal to the beam axis, remain normal after bending. With this we can define stress-strain variation through the thickness as:
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k = \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix}_k \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o
\end{bmatrix} + z \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}_k
\] (7)

The \( \overline{Q}_{ij} \) can be different for each layer due to material or angle differences. From this we can get moments and forces per unit length:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix}_k \begin{bmatrix}
z_k \\
z_k \\
z_k
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o
\end{bmatrix} + \int_{z_k}^{z_{k+1}} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} zdz
\] (8)

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix}_k \begin{bmatrix}
z_k \\
z_k \\
z_k
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o
\end{bmatrix} + \int_{z_k}^{z_{k+1}} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} z^2dz
\] (9)

where \( N \) is the number of layers in the laminate. Since the layers create a non-continuous curve, we can change the integrals to a piecewise function. We can then turn (8) and (9) into a matrix form:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix} \begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o
\end{bmatrix}_k
\] (10)

where,

\[
A_{ij} = \sum_{k=1}^{N} \overline{Q}_{ij} \left( z_k - z_{k-1} \right)
\] (11)

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij} \left( z_k^2 - z_{k-1}^2 \right)
\] (12)

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij} \left( z_k^3 - z_{k-1}^3 \right)
\] (13)

This set can be rewritten in a more compact form:
Using this information, we can get the effective stiffnesses. Effective stiffnesses are derived by taking the force-moment equations and matching them with the flexural and torsion formulas. For our purpose we are looking into the bending stiffness thus we first get the mid plane strains and curvatures as functions of forces and moments per unit length:

\[
\begin{bmatrix} \varepsilon^o \\ \kappa \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix}
\]  

(15)

For symmetric laminates, \( B = 0 \) and it then can be shown \( A^* = A^{-1} \) and \( D^* = D^{-1} \). This reduces equation (15) to:

\[
\begin{bmatrix} \varepsilon^o \\ \kappa \end{bmatrix} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix}
\]  

(16)

We can then solve for the effective stiffnesses by setting certain forces to zero in (16). For flexural effective Young’s modulus, we can set forces and moments \( M_y \) and \( M_{xy} \) all equal to zero, which leads to the matrix:

\[
\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} M_x \\ 0 \\ 0 \end{bmatrix}
\]  

(17)

Solving the first equation of (17) for \( \kappa_x \):

\[
\kappa_x = D_{11}^* M_x
\]  

(18)

Equation (18) can then be applied to the beam’s flexural formula:

\[
E_x I_x \kappa_x = M_x
\]  

(19)

\( M_x \) in (18) is per unit length thus we multiply by beam width, \( b \):

\[
\kappa_x = D_{11}^* M_x b
\]  

(20)
Solving for flexural effective Young’s modulus in (19) and plugging in \( I_x \) for a rectangle cross section beam and (20), the effective Young’s modulus becomes:

\[
E_{x}^{f} = \frac{12}{h^3 D_{11}}
\]  

(21)

We can then do a similar formulation for the effective flexural shear modulus. This is completed by setting all forces, \( M_x \) and \( M_y \) to zero. Doing a similar process as before we get \( \kappa_{xy} \):

\[
\kappa_{xy} = D_{66} M_{xy}
\]

(22)

We then use the beam’s torsion formula:

\[
G_{xy}^f J \kappa_{xy} = M_{xy}
\]

(23)

Putting together all variables and solving for \( G_{xy}^f \):

\[
G_{xy}^f = \frac{12}{h^3 D_{66}}
\]

(24)

1.7 - Weak formulation for a Beam Finite Element

To start the formulation of the motion of the beam, we investigate the minimum potential energy theorem which is written as:

\[
\Pi = U + T + W
\]

(25)

where \( U \) is the strain energy, \( T \) is the kinetic energy and \( W \) is the work done.

1.7.1 - Shape Functions: Beam Bending

Shape functions are derived using polynomial functions:

\[
v = a_0 + a_1 x + a_2 x^2 + a_3 x^3
\]

(26)

\[
\theta = \frac{dv}{dx} = a_1 + 2a_2 x + 3a_3 x^2
\]

(27)

using boundary conditions:

\[
v(x = 0) = v_1 = a_0
\]
\[ \theta(x=0) = \theta_1 = a_1 \]
\[ v(x=L) = v_2 = a_0 + a_1L + a_2L^2 + a_3L^3 \]
\[ \theta(x=L) = \theta_2 = a_1 + 2a_2L + 3a_3L^2 \]

rewriting this in matrix form:

\[
\begin{bmatrix}
\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & L & L^2 & L^3 \\
0 & 1 & 2L & 3L^2
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3
\end{bmatrix}
\]
\]

(28)

since \( \theta \) is a function of \( v \), we can solve for the \( a \) coefficients in (28) and plug into (26)

\[
v = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{3}{L^2} & -\frac{2}{L} & -\frac{3}{L^2} & \frac{1}{L} \\
\frac{2}{L^2} & \frac{1}{L} & \frac{2}{L^2} & \frac{1}{L^2}
\end{bmatrix}
\begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\
\end{bmatrix}
\]

(29)

Expanding (29) and replacing \( x \) with \( \xi = \frac{2x}{L} - 1 \), to change limits from 0 \( \rightarrow \) \( L \) to -1 \( \rightarrow \) 1,

\[
v = \frac{1}{4}(1-\xi)^3(2+\xi)v_1 + \frac{1}{8}L(1-\xi)^2(1+\xi)\theta_1
\]
\[
-\frac{1}{4}(1+\xi)^3(2-\xi)v_2 - \frac{1}{8}L(1+\xi)^2(1-\xi)\theta_1
\]

(30)

from inspection of (30) we get shape functions

\[
N_{v1} = \frac{1}{4}(1-\xi)^3(2+\xi)
\]
\[
N_{\theta1} = \frac{1}{8}L(1-\xi)^2(1+\xi)
\]
\[
N_{v2} = \frac{1}{4}(1+\xi)^3(2-\xi)
\]
\[
N_{\theta2} = \frac{1}{8}L(1+\xi)^2(1-\xi)
\]

(31)
Then

\[ v = \begin{bmatrix} N_{v1} & N_{\theta1} & N_{v2} & N_{\theta2} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \]  

(32)

which can be written in compact form as \( v(\xi) = \mathbf{N}\mathbf{d} \).

1.7.2 - Beam Stiffness

In solid mechanics one can look at the total potential energy of a system being zero. Solving for the strain energy of an element:

\[ U_v = \frac{1}{2} \int \sigma \varepsilon \, dV \]  

(33)

For a beam element the nodal displacement vector is:

\[ \mathbf{u}^t = \begin{bmatrix} v_1 & \theta_1 \\ v_2 & \theta_2 \end{bmatrix} \]  

(34)

\[ \varepsilon = \frac{du}{dx} = -y \frac{d^2v}{dx^2} = -y\kappa \]  

(35)

where \( \kappa \) is the curvature of the beam about its span. The bending stress is written as:

\[ \sigma = E\varepsilon = -Ey\kappa \]  

(36)

With this information we can return to equation (33) and get the general formulation for potential energy:

\[ U_v = \frac{1}{2} \int_{x A} \sigma \varepsilon A \, dx = \frac{1}{2} \int_{x A} (-Ey\kappa)(-y\kappa) \, dA \, dx \]

\[ = \frac{1}{2} \int_{x} E\kappa^2 \int_{A} y^2 \, dA = \frac{1}{2} \int_{0}^{L} EI\kappa^2 \, dx = \frac{1}{2} \int_{0}^{L} EI \left( \frac{d^2v}{dx^2} \right)^2 \, dx \]

\[ U_v = \frac{1}{2} \int_{0}^{L} \nu^2 EIv'' \, dx \]  

(37)

(38)

The curvature can then be rewritten as a function of \( \xi \):

\[ \kappa = \text{function of } \xi \]
\[ \kappa = \frac{d^2v(x)}{dx^2} = \frac{4}{L^2} \frac{d^2v(\xi)}{d\xi^2} = \frac{4}{L^2} \frac{d^2\bar{N}}{d\xi^2} \bar{d} = B\bar{d} \]  \hspace{1cm} (39)

where \( \bar{N} \) is a matrix containing shape functions and \( \bar{d} \) is a vector of nodal displacements.

The strain energy can then be rewritten:

\[ U_e = \frac{1}{2} \int_0^L v''EIv'' \, dx = \frac{1}{2} \left( \bar{d}^T \bar{B}^T \right) EI \left( \bar{B} \bar{d} \right) Ld\xi = \bar{d}^T \left( \frac{1}{2} \int_{-1}^1 \bar{B}^T EI\bar{B} \, d\xi \right) \bar{d} \]  \hspace{1cm} (40)

From (40), stiffness can be obtained as:

\[ K = \frac{1}{2} \int_{-1}^1 \bar{B}^T EI\bar{B} \, d\xi \]  \hspace{1cm} (41)

1.7.3 - Beam Mass

The mass matrix for a beam is like a bar. The derivation starts with the kinetic energy of the element expressed as:

\[ T = \frac{1}{2} \int_0^L m \left( \frac{du(x,t)}{dt} \right)^2 dx = \bar{u}^T \left( \frac{1}{2} \int_0^L mN^T dx \right) \bar{u} = \bar{u}^T M\bar{u} \]  \hspace{1cm} (42)

From (42), one can show the mass matrix as:

\[ M = \frac{1}{2} \int_0^L mN^T dx \]  \hspace{1cm} (43)

Similar to the stiffness matrix we change the limits of \( x \) from 0 \( \rightarrow \) \( L \) to -1 \( \rightarrow \) 1, thus (43) is changed to:

\[ M = \frac{1}{2} \int_{-1}^1 mN^T \, d\xi \]  \hspace{1cm} (44)

1.7.4 - Shape Functions: Shaft

\[ \phi = a_0 + a_i x \]  \hspace{1cm} (45)

\[ \phi = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \]  \hspace{1cm} (46)
Applying boundary conditions to (46):

\[
\phi(x = 0) = \phi_1 = a_0 \quad (47)
\]

\[
\phi(x = L) = \phi_2 = a_0 + a_1L \quad (48)
\]

\[
\begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
1 & L
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1
\end{bmatrix} \quad (49)
\]

\[
\begin{bmatrix}
a_0 \\
a_1
\end{bmatrix} =
\frac{1}{L}
\begin{bmatrix}
L & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} \quad (50)
\]

We can then utilize the solution for \( a_o \) and \( a_i \) in (46):

\[
\phi = \frac{1}{L} [1 \ x]
\begin{bmatrix}
L & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} \quad (51)
\]

In order to use the gauss-quadrature we must have out limits go from \( x = 0 \rightarrow L \) to \( \xi = -1 \rightarrow 1 \) using the transformation:

\[
\xi = \frac{2x - L}{L} \quad (52)
\]

Replacing \( x \) with \( \xi \) :

\[
\phi = \frac{1}{L}
\begin{bmatrix}
1 \\
\frac{L(\xi + 1)}{2}
\end{bmatrix}
\begin{bmatrix}
L & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} \quad (53)
\]

\[
\phi = \frac{1}{L}
\begin{bmatrix}
\frac{(\xi + 1)}{2} \\
L & 0 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} \quad (54)
\]

\[
\phi = \frac{1}{L}
\begin{bmatrix}
\frac{(1-\xi)}{2} \\
\frac{(\xi + 1)}{2}
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix} \quad (55)
\]

From inspection of (55) we can get the shape functions for the shaft:

\[
N_{i1} = \frac{1 - \xi}{2} \quad (56)
\]

\[
N_{i2} = \frac{\xi + 1}{2} \quad (57)
\]

27
(52) also leads to a derivative:

\[ \frac{d\xi}{dx} = \frac{2}{L} \]  \hspace{1cm} (58)

1.7.5 - Shaft Stiffness

Like the beam stiffness we can start with potential energy of a rotating shaft:

\[ U_e = \frac{1}{2} \int \tau \gamma dV \]  \hspace{1cm} (59)

where,

\[ \tau = \frac{Tr}{J} \]  \hspace{1cm} (60)
\[ \gamma = r \frac{d\phi}{dx} \]  \hspace{1cm} (61)

We can now plug (60) through (61) into (59), thus leading to:

\[ U_e = \frac{1}{2} \int \phi^T \frac{d\phi}{dx} \frac{d\phi}{dx} = \frac{1}{2} \int \frac{L}{2} GJ \left( \frac{d\phi}{dx} \frac{d\phi}{dx} \right) d\xi \]  \hspace{1cm} (62)

\[ = \frac{1}{2} \int \frac{L}{2} GJ \phi^T B^T B \phi d\xi \]  \hspace{1cm} (63)

From (63), we can get the stiffness matrix that is identical to (41), with just changed material and geometrical properties, \( G \) and \( J \):

\[ K = \frac{1}{2} \frac{L}{2} GJ \int B^T B \phi d\xi \]  \hspace{1cm} (64)

The mass moment of inertia for a rectangular beam can be used:

\[ J = \frac{DLhb}{12} \left( h^2 + b^2 \right) \]  \hspace{1cm} (65)

1.7.6 - Shaft Mass

Using almost the same method for a rotating shaft as a beam element, we can write the kinetic energy as:
\[
T = \frac{1}{2} \int_{0}^{L} \left( \frac{d\phi(x,t)}{dx} \right)^2 \ dx = \dot{\Theta}^T \left( \frac{1}{2} \int_{0}^{L} JN^T \ d\xi \right) \dot{\Theta} = \dot{\Theta}^T \ M \dot{\Theta}
\]  
(66)

Shift the limits to \(-1 \rightarrow 1\) in (66) we can get the mass matrix for a rotating shaft:

\[
M = \frac{JL}{2} \int_{-1}^{1} NN^T \ d\xi
\]  
(67)

1.8 - Dynamic Stiffness and Transfer Matrix

For periodic analysis we must define what is known as the transfer matrix to find our stop bands. Like [61] we start with the dynamic equation of motion for the first segment of the cell.

\[
\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]  
(68)

where \( u_i = \{v_i, \theta_i\}^T, i,j = 1-2 \), is the displacement and rotation at node \( i \) and \( f_i = \{f_{v_i}, M_{\theta_i}\}^T \) is a force vector including the force and moment at node \( i \). Then \( m_{ij} \) and \( k_{ij} \) represent the terms of the mass and stiffness matrices, respectively. In harmonic excitation we can assume a solution for force and displacement:

\[
\begin{align*}
\mathbf{u}_i &= \mathbf{\bar{u}}_i e^{i\omega t} \\
\mathbf{f}_i &= \mathbf{\bar{f}}_i e^{i\omega t}
\end{align*}
\]  
(69)(70)

Accordingly, we can get the acceleration as:

\[
\ddot{\mathbf{u}}_i = -\omega^2 \mathbf{u}_i
\]  
(71)

Putting this back in (68), we get

\[
\begin{bmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]  
(72)

We will define the matrix in-front of the displacement as \( \mathbf{D} \), the “dynamic stiffness matrix”:

\[
\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} \end{bmatrix}
\]  
(73)
With this (72) changes to a new form.

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]  (74)

Solving the first row of (74) for \( u_2 \) we get

\[
u_2 = -D_{12}^{-1}D_{11}u_1 + D_{12}^{-1}f_1
\]  (75)

This can now be substituted in the second row of (74) to get

\[
f_2 = (D_{21} - D_{12}^{-1}D_{11}u_1)u_1 + D_{22}D_{12}^{-1}f_1
\]  (76)

Rewriting (75) and (76) in matrix form leads to

\[
\begin{bmatrix}
-D_{12}^{-1}D_{11} & D_{12}^{-1} \\
D_{21} - D_{12}^{-1}D_{11}u_1 & D_{22}D_{12}^{-1}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
f_1
\end{bmatrix} =
\begin{bmatrix}
u_2 \\
f_2
\end{bmatrix}
\]  (77)

The same setup and derivations can then be done for the second segment of the cell.

\[
\begin{bmatrix}
-D_{12}^{-1}D_{11} & D_{12}^{-1} \\
D_{21} - D_{12}^{-1}D_{11}u_1 & D_{22}D_{12}^{-1}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
f_2
\end{bmatrix} =
\begin{bmatrix}
u_3 \\
f_3
\end{bmatrix}
\]  (78)

Considering displacement continuity, \( u_3 \) is equal to \( u_1 \) of the following cell and from equal and opposite equilibrium forces, \( f_3 \) is equal to \(-f_1\), the negative sign indicating opposite directions of forces and moments. We can write a relationship between \( f_3 \) and \( u_3 \) on the right-hand of the cell and \( f_1 \) and \( u_1 \) on the left-hand side as:

\[
\begin{bmatrix}
u_3 \\
f_3
\end{bmatrix} = \lambda \begin{bmatrix}
u_1 \\
f_1
\end{bmatrix}
\]  (79)

where \( \lambda \) is a constant. (79) can then be combined with (78) giving

\[
\begin{bmatrix}
-D_{12}^{-1}D_{11} & D_{12}^{-1} \\
D_{21} - D_{12}^{-1}D_{11}u_1 & D_{22}D_{12}^{-1}
\end{bmatrix}
\begin{bmatrix}
u_2 \\
f_2
\end{bmatrix} = \lambda \begin{bmatrix}
u_1 \\
f_1
\end{bmatrix}
\]  (80)

Then combining (80) with (77) gives a eigenvalue matrix problem shown as
\[
\begin{bmatrix}
-D_{12} D_{11} & D_{12} \\
D_{21} - D_{12} D_{11} u_1 & D_{22} D_{11}^{-1}
\end{bmatrix}_{\text{Segment 1}}
\begin{bmatrix}
-D_{12} D_{11} & D_{12} \\
D_{21} - D_{12} D_{11} u_1 & D_{22} D_{11}^{-1}
\end{bmatrix}_{\text{Segment 2}}
\begin{bmatrix}
u_1 \\
f_1
\end{bmatrix} = \lambda \begin{bmatrix}
u_1 \\
f_1
\end{bmatrix}
\]

The matrix multiplication we then be represented as \( T \).

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
f_1
\end{bmatrix} = \lambda \begin{bmatrix}
u_1 \\
f_1
\end{bmatrix}
\] (82)

\( T \) is called the “transfer-matrix of the cell.” The transfer matrix allows us to then define \( \lambda = e^{-\mu} \) where \( \mu \) is called the “propagation factor”. From inspection, (82) is an eigenvalue problem where,

\[
e^{\mu} = \text{eigenvalue} \begin{bmatrix}
T_{11} & T_{11} \\
T_{12} & T_{22}
\end{bmatrix}
\] (83)

The transfer matrix is derived from the dynamic stiffness matrix. The dynamic stiffness matrix is a function of the excitation frequency \( \omega \), then \( \mu \) is a function of frequency. The eigenvalues of the transfer matrix can be shown to come in pairs, one being the reciprocal of the other. The propagation factor can then be obtained from:

\[
\mu = \cosh^{-1}\left( \frac{e^{\mu} + e^{-\mu}}{2} \right) = \alpha + i\theta
\] (84)

where \( \alpha \) is the attenuation of the vibration amplitude and \( \theta \) represents the phase difference between the input and output waves. When plotting the propagation factor on a \( \mu \) verses \( \omega \) graph, stop bands can be seen. Overlaying these onto a frequency response plot of non-periodic and periodic beams we can see the exact shifts of natural frequencies.

The attenuation of amplitudes is found when looking deeper into \( \alpha \). When putting (84) in (83) we get \( e^{\alpha + i\theta} = e^{\alpha} e^{i\theta} \). \( \alpha \) can become positive or negative. When \( \alpha \) is negative, \( e^{\alpha} < 1 \) causes attenuation and when \( \alpha \) is positive, \( e^{-\alpha} < 1 \) causes attenuation.

Plotting \( \alpha \), we see that when \( \alpha = 0 \), there is no attenuation, this location is classified as the passband region. When \( \alpha \neq 0 \), there is an indicated stopband, where waves are attenuated. In Figure 18, we can see the attenuation factor plotted on the right axis. This is plotted along with the FRP of a plain and periodic beam to show the reduction of natural frequencies in the stop band regions.
Figure 18: Graph to show representation of attenuation factor plotted over a plain and periodic beam.
2.1 - Model Description

A model of a beam was created in SOLIDWORKS as a surface feature. This was comprised of multiple sections to emulate a periodic beam itself, seen in Figure 19. Each section was made as a surface, then patterned to be able to change number of patches, PSR, length of the beam and width of the beam. Simulations were done using shell elements and the composite feature in the material selection area.

Figure 19: SOLIDWORKS mode made with surface patterns

2.2 - Convergence

Using a beam of equal size as the one to be worked with in other studies, convergence of the frequencies was examined. To make sure the solution is, numerically, as close to the exact solution the following convergence criteria was used:

\[ \% \text{diff} = \left( \frac{f_{\text{previous}} - f_{\text{current}}}{f_{\text{current}}} \right) \times 100 \]  

Figure 20 shows the first mode frequency vs mesh size. In this figure as the mesh size decreases, the modal frequency approaches the exact solution. As we get to a mesh size of \(1/h = 160\), the relative error is 0.3\. Due to this, a mesh of 0.004 was used.

Figure 20: Convergence plot of first mode frequency
2.3 - Verification

In order to utilize SOLIDWORKS as a foundation to relate other results to, studies comparing other materials and plotting their natural frequencies were done. The studies were referencing isotropic beams then comparing an analytical composite beam solution. After this was done, comparison of a carbon fiber-reinforced laminated beam was done to verify SOLIDWORKS composite shell frequency analysis.

2.3.1 - Isotropic Beam

As a basis of computation, an isotropic beam in a cantilever configuration was analyzed and compared to an analytical solution for multiple natural frequencies as shown in tables 1 and 2.

*Table 1: Comparison of 6061 Aluminum cantilever beam to analytical solution*

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOLIDWORKS [Hz]</th>
<th>Analytical [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>51.396</td>
<td>51.401</td>
<td>0.011</td>
</tr>
<tr>
<td>2</td>
<td>321.28</td>
<td>322.153</td>
<td>0.271</td>
</tr>
<tr>
<td>3</td>
<td>897.1</td>
<td>909.033</td>
<td>1.312</td>
</tr>
</tbody>
</table>

*Table 2: Comparison of ANSI 1020 Steel cantilever beam to analytical solution*

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOLIDWORKS [Hz]</th>
<th>Analytical [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>51.071</td>
<td>50.794</td>
<td>0.545</td>
</tr>
<tr>
<td>2</td>
<td>319.28</td>
<td>318.344</td>
<td>0.293</td>
</tr>
<tr>
<td>3</td>
<td>891.4</td>
<td>898.285</td>
<td>0.766</td>
</tr>
</tbody>
</table>

2.3.2 - Composite Beam

In order to make sure SOLIDWORKS is a viable option for composite beam finite element analysis, a composite beam was created and supported in a simply supported configuration. The beam is defined with a length to height ratio (L/h) of 120. The height of which is 0.00635 meters.

The material properties relative to global coordinates necessary for composite analysis in SOLIDWORKS are $E_x = 144.84$ GPa, $E_y = E_z = 9.65$ GPa, $\nu_{xy} = \nu_{xz} = 0.3$, and $\rho = 1389.79$ kg/m$^3$. Table 3 shows the comparison. The difference never exceeded 1.5%.

34
Table 3: Comparison of natural frequencies of a simply supported 0-degree composite beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOLIDWORKS [Hz]</th>
<th>Analytical [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.565</td>
<td>51</td>
<td>0.852</td>
</tr>
<tr>
<td>2</td>
<td>201.55</td>
<td>202</td>
<td>0.222</td>
</tr>
<tr>
<td>3</td>
<td>450.83</td>
<td>453</td>
<td>0.479</td>
</tr>
<tr>
<td>4</td>
<td>795.04</td>
<td>802</td>
<td>0.867</td>
</tr>
<tr>
<td>5</td>
<td>1229.6</td>
<td>1248</td>
<td>1.474</td>
</tr>
</tbody>
</table>

2.4 - Sweep Analysis

2.4.1 - Effect of Periodic Ply Angle

As seen in Figure 21, as the ratio of ply angle in the periodic patch increases there is an apparent decrease in frequency, but as this ratio goes up, the ply angle will have a less effect on the frequency. This is apparent when looking at all modes, including those which increase in frequency instead of decreasing, such as the third mode shown in Figure 22.

Figure 21: First and Second Mode frequency vs periodic ply angle
2.5 - Clamp Vibration Analysis

When looking into a clamp to use for vibrational testing, certain things such as form factor and modal issues arose. The form of the clamp allows for beams to be placed in such a way that they will align with the shaker and stringer. The modal issues are in respect to a possible mode of the test setup affecting the beams output response. Following are the design and modal analysis to make sure these issues are accounted for.

2.5.1 - Design of the Clamp

The design of the clamp, shown in Figure 23, was a simple one that allowed for clamps to be placed on the bottom and a piece of aluminum to be bolted to the side. The bolted material would allow for a beam to be placed in between to be clamped in a cantilever configuration. This will be shown in more detail later.
2.5.2 - Clamp Modal Analysis

To make sure the clamps modal frequencies would not interfere with the experiment, a frequency analysis was completed on the clamp in SOLIDWORKS to see where its natural frequencies were. This allows, if needed, a design change to be made before making the clamp. A fixed support was placed on the bottom face of the clamp. The modes can be seen in Table 4. Figure 24 shows the second mode of vibration which is a twisting mode.

Table 4: Modal Frequencies and Shapes of clamp

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency [Hz]</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>822.06</td>
<td>Bending</td>
</tr>
<tr>
<td>2</td>
<td>1963.1</td>
<td>Twist</td>
</tr>
<tr>
<td>3</td>
<td>4547.5</td>
<td>Bend</td>
</tr>
</tbody>
</table>

These frequencies are very high, most measurements will be taken up to 800 Hz. The second mode is the most pertinent to the testing. As shown in Figure 24, the twist is in a direction that could be induced by the excitation of the beam bending. The natural frequency for this is nearly 2000 Hz, so we now know we have an upper bound on the testing and the design does not need any changes.
2.6 - Discussion

With the use of SOLIDWORKS, it can be seen that good results for isotropic and composite materials were obtained. Analysis can be done on periodic beams using surface features and the composite function in the frequency analysis toolset. It has adequate results when compared to analytical solutions.

Although SOLIDWORKS can be used as a general modal analysis tool, it does not have a function for periodic analysis or frequency response plots of composite materials. It can be seen with the analysis done here that there is a limit to how many plies one can add to the periodic patches before the effect of the angles of the plies change the natural frequencies themselves.

As a design tool, SOLIDWORKS, allows for a large range of analysis tools. This allows the modal analysis of the created clamp base to make sure unexpected natural frequencies are easier to detect. The modal analysis allowed for a shape to be found that would affect the results of the vibration testing. This shape’s frequency was found to be much higher than any frequency sweeps done experimentally, thus no change in the clamp was needed to be done.
Chapter 3 - In-House FE Vibration Analysis

3.1 - Model Description

Utilizing a composite beam model, stop bands can be determined for beams. Several studies were done looking into how certain parameters affect the stop bands. The properties to vary are number of periodic layers, NPL, angle of periodic layers, APL, and periodic layer ratio, PLR. All the parameters we can vary are specific for composite structures since they are created by layering plies and these layered plies can have varied angles and relative sizing.

Each parameter has its own meaning. PLM, periodic layer material, allows one to change the material of each ply of the periodic section. NPL, number of periodic layers, can vary from one to infinity. It is numbered based on how many layers are placed on either side of the main beam. For example, if NPL is two there is two layers on each side of the beam. PPA, periodic ply angles, can vary from zero to one-hundred-and-eighty. We limit this to zero through ninety, since all angles passed ninety give the same results as their counterparts. An example of this would be properties of a composite with one-hundred-degree ply’s is the same as the same composite with eighty-degree plies. Finally, PSR, periodic segment ratio, is a ratio of cell length to periodic segment, $PSR = L_2/(L_1+L_2)$, as shown in Figure 25 PSR is varied from 0.1 to 0.9 due to ratios less than this would be impractical to be physically produced.

![Figure 25: Dimensions and nomenclature for periodic beam with composites](image)

The main beam can also vary in its parameters. It has its own respective material, ply angle, and number of plies. These parameters are all the same as the periodic section but apply for the whole length of the main beam.
3.2 - Model Verification

Using a beam of equal size as the one to be worked with in other studies, convergence of the frequencies was done. To make sure the solution is numerically as close to the analytical solution the following convergence measure was used:

\[ \%Diff = \left( \frac{\omega_{\text{analytical}} - \omega_{\text{fem}}}{\omega_{\text{analytical}}} \right) \times 100 \]

To find the difference between the numerical code produced and the analytical solution, we created a beam with 10 elements (5 cells) and compared its results to the analytical approach. This difference is shown in Table 5 below.

*Table 5: Analytical vs numerical modal frequencies for cantilever carbon fiber composite beam.*

<table>
<thead>
<tr>
<th>Mode</th>
<th>Analytical Freq [Hz]</th>
<th>Numerical Freq [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>51</td>
<td>49.6</td>
<td>2.75%</td>
</tr>
<tr>
<td>2</td>
<td>203</td>
<td>198.6</td>
<td>2.17%</td>
</tr>
<tr>
<td>3</td>
<td>457</td>
<td>447</td>
<td>2.19%</td>
</tr>
<tr>
<td>4</td>
<td>812</td>
<td>795.6</td>
<td>2.02%</td>
</tr>
<tr>
<td>5</td>
<td>1269</td>
<td>1245.9</td>
<td>1.82%</td>
</tr>
</tbody>
</table>

As a secondary verification of the numerical model, a periodic beam was created and analyzed as shown in Table 6 and Table 7 and compared to SOLIDWORKS. Changing any other parameter never caused a difference larger than 9% compared to SOLIDWORKS. This further solidifies our confidence in the developed numerical model.

*Table 6: Cantilever E-glass/epoxy, length 0.4m, width 0.05m, NPL 2, NML 2, PSR 0.4, NC 5, PPA 0*

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOLIDWORKS [Hz]</th>
<th>Numerical Freq [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.779</td>
<td>0.754</td>
<td>3.28</td>
</tr>
<tr>
<td>2</td>
<td>4.986</td>
<td>4.823</td>
<td>3.265</td>
</tr>
<tr>
<td>3</td>
<td>14.366</td>
<td>13.901</td>
<td>3.237</td>
</tr>
<tr>
<td>4</td>
<td>29.364</td>
<td>28.434</td>
<td>3.165</td>
</tr>
</tbody>
</table>
Table 7: Cantilever E-glass/epoxy, length 0.4m, width 0.05m, NPL 2, NML 2, PSR 0.4, NC 5, PPA 30

<table>
<thead>
<tr>
<th>Mode</th>
<th>SOLIDWORKS [Hz]</th>
<th>Numerical Freq [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.715</td>
<td>0.746</td>
<td>4.26</td>
</tr>
<tr>
<td>2</td>
<td>4.528</td>
<td>4.763</td>
<td>5.19</td>
</tr>
<tr>
<td>3</td>
<td>12.890</td>
<td>13.699</td>
<td>6.276</td>
</tr>
<tr>
<td>4</td>
<td>25.826</td>
<td>27.937</td>
<td>8.173</td>
</tr>
</tbody>
</table>

3.3 - Sensitivity Analysis

Sweeps were conducted looking into a carbon fiber-reinforced beam with carbon fiber-reinforced periodic patches, and 0.2 mm ply thickness. Material properties used for the effective stiffness calculations were, $E_1 = 144.7$ GPa, $E_2 = 9.652$ GPa, $\nu_{12} = 0.33$, $G_{12} = 4.136$ GPa and $\rho = 1389.9$ Kg/m$^3$. The dimensions of the beam are as follows unless swept through: the main beam has a length of 40 cm and width of 5 cm, the base beam has 4 plies and periodic plies also have 4 plies, main ply angle is 0 degree, PPA = 0 degrees, PSR = 0.7, NC = 5.

3.3.1 - Effect of PSR on Stopbands

Effects due to length of periodic patches was the first sweep done. Figure 26 and Figure 27 show that with increasing PSR the first stop band will slightly decrease in size then significantly increase, while the second will decrease in size. There is a shift in the location of the stop band after PSR = 0.6 for the first stop band and PSR = 0.5 in the second stopband.

![Figure 26: Effect of PSR on first stop band.](image)
Once a general trend was seen for the effect of PSR, how each of the other parameters affect the size of the stopbands was investigated. Figure 28 and Figure 29 show how the stopbands are changed by NC. Increasing NC will shift the stop bands to higher frequencies and also makes them wider.

The effect of changing NPL in addition to PSR is shown in Figure 30 and Figure 31. The effect on the first stop band is different from that on the second. As NPL increases, the size of the first stop band increases when PSR is between 0.1 and 0.3 or larger than 0.8. The size of the second stop band increases as NPL increases for all values of PSR. The frequency shift of the atop bands is not as significant as in the case of sweeping NC. The effect of changing PPA is also shown in Figure
32 and Figure 33. Increasing PPA results in decreasing the size of both stop bands and shifts them to lower frequencies.

![Image](image1.png)  ![Image](image2.png)

*Figure 30: First stopband, sweeping PSR while changing NPL*

*Figure 31: Second stopband, sweeping PSR while changing NPL*

![Image](image3.png)  ![Image](image4.png)

*Figure 32: First stopband, sweeping PSR while changing PPA*

*Figure 33: Second stopband, sweeping PSR while changing PPA*

3.3.2 - **Effect of PPA on Stopbands**

After the PSR was swept, the next sweep was of PPA. Increasing the angle of the periodic plies results in decreasing the size of the first stop band significantly as shown in Figure 34. It also shifted the location of the second stop band to lower frequencies without a significant effect on the size past 30° as shown in Figure 35.
Investigating how other parameters change the stopbands can be seen below. Figure 36 through Figure 41 show that other parameters do not have large effects on the trend of the decrease of the size of stopbands as PPA increases. Increasing NC in Figure 36 and Figure 37 shifts both stop bands to higher frequencies and increases their size. Increasing NPL in Figure 38 and Figure 39 mainly increases the size of both stop bands. Increasing PSR in Figure 40 increases the size of the first stop band if PPA is less than 45° but slightly decreases the size past 45°. Increasing PSR also decreases the size of the second stop band as shown in Figure 41. All sweep studies reported in Figure 36 through Figure 41 also show that the size and location of both stop bands remain constant past 45° PPA.
Figure 36: First stopband, sweeping PPA while changing NC

Figure 37: Second stopband, sweeping PPA while changing NC

Figure 38: First stopband, sweeping PPA while changing NPL

Figure 39: Second stopband, sweeping PPA while changing NPL
3.3.3 - Effect of NPL on Stopbands

The effect of NPL on the first two stop bands is shown in Figure 42 and Figure 43. The first stop band is seen to shift to lower frequencies as NPL increases without great change in size. The size of the second stop band, however, significantly increases as NPL increases in Figure 43.
When changing other parameters and sweeping NPL, in Figure 44 to Figure 49, similar trends can be seen as in the previous studies. Increasing NC always increases the size of the stop bands and shifts them to higher frequencies (Figure 44 and Figure 45). Increasing PPA up to 45° decreases the size of the stop bands and shifts them to lower frequencies. However, the stop bands are not significantly affected past 45° PPA (Figure 46 and Figure 47). Increasing PSR significantly increases the size of the first stop band but decreases that of the second (Figure 48 and Figure 49).
3.3.4 - Effect of NC on Stopbands

Finally we can look at the number of cells (NC) parameter. Figure 50 and Figure 51 show the effect of NC on the first two stop bands. As NC increases the size of both stop bands increases and they shift to significantly higher frequencies. The first stop band of a periodic beam with larger NC could be at the same location of the second stop band of a periodic beam with same characteristics but with lower NC.
Figure 50: First stop band with sweeping number of cells (NC)

Figure 51: First two stop bands with sweeping number of cells (NC)

After a NC was swept as a single parameter, other parameters were then swept. During all the changes to the parameters, the trend of increasing size of both stopbands were the same for Figure 52 through Figure 57. All previous observations can be seen in these figures as well. Increasing NPL increases the size of the second stop band (Figure 52 and Figure 53). Increasing PPA decreases the sizes of both stop bands only up to 45°, then no effects is seen (Figure 54 and Figure 55). Increasing PSR increases the size of the first stop band but decreases that of the second (Figure 56 and Figure 57).
Figure 52: First stopband, sweeping NC while changing NPL

Figure 53: Second stopband, sweeping NC while changing NPL

Figure 54: First stopband, sweeping NC while changing PPA

Figure 55: First stopband, sweeping NC while changing PPA
3.4 - One Dimensional Rectangular Cross-Sectional Shafts

Due to the fact that non-circular cross sections do not have constant shear strain, the formulation for their stiffness matrix becomes long and commonly entails the use of St. Venant’s principle and turning the element into a 2D element. To reduce the dimensions to 1D, a simple study was done to relate a non-circular cross section to a circular one and create a multiplication factor to the stiffness matrix.

If we assume that the natural frequencies are to be the same but with a factor attached to it, we can write the circle’s natural stiffness, subscript “c”, and non-circular stiffness, subscript “nc”, as:

\[ K_c = \omega_c^2 m \]
\[ K_{nc} = \omega_{nc}^2 m \]

Assuming that the mass matrix is the same for each beam, we can divide (86) and (87). This gets an \( \alpha \) factor between the two stiffnesses:

\[ \alpha = \frac{K_{nc}}{K_c} = \frac{\omega_{nc}^2}{\omega_c^2} \]

Where \( \omega_{nc} \) is the first natural frequency from SOLIDWORKS of a rectangular cross section and \( \omega_c \) is found using the analytical solution for a 1m wide circular cantilever beam:
\[ \omega_c = \frac{\pi}{L} \sqrt{\frac{G}{\rho}} \quad (89) \]

Data was then collected, starting with a square cross section 6061 aluminum beam of 1m \( \times \) 1 m \( \times \) 12m and then reducing the height of the beam until an extremely thin section was created. A common thickness of composite is 0.2 mm thick, thus a 0.0004 height-to-width ratio was the lower limit considering beams in these studies do not get that low of ratio. \( \alpha \) was then graphed relative to the height-to-width ratio. Then MATLAB’s built-in function, Polyfit, was used to find an appropriate equation to fit the same curve. The curve and fit curve of the 3\(^{\text{rd}}\) order can be seen in Figure 58.

![Graph showing \( \alpha \) vs height-to-width ratio of rectangular beam](image)

*Figure 58: Alpha vs height-to-width ratio of rectangular beam*

3.4.1 - *Verification*

To verify the utilization of a torsion factor, an isotropic beam and composite beam were analyzed in SOLIDWORKS, then the same beam was analyzed using MATLAB in-house code. The isotropic beam in question was 0.4 m in length with a 0.05m width. The heights of the beam were changed and listed with two materials, ANSI 1020 and 6061 Aluminum alloy, in Table 8 and Table 9, respectively.
Table 8: First twist frequency of ANSI 1020 steel beam with rectangular CS and sweeping height

<table>
<thead>
<tr>
<th>h</th>
<th>SOLIDWORKS [rad/sec]</th>
<th>MATLAB [rad/sec]</th>
<th>%diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>2425.8</td>
<td>2465.7</td>
<td>1.644818</td>
</tr>
<tr>
<td>0.002</td>
<td>926.536</td>
<td>976.6546</td>
<td>5.409245</td>
</tr>
<tr>
<td>0.001</td>
<td>503.78</td>
<td>476.45</td>
<td>5.424987</td>
</tr>
</tbody>
</table>

Table 9: First twist frequency of 6061 Al. Alloy beam with rectangular CS and sweeping height

<table>
<thead>
<tr>
<th>h</th>
<th>SOLIDWORKS [rad/sec]</th>
<th>MATLAB [rad/sec]</th>
<th>%diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>2412.9</td>
<td>2450.8</td>
<td>1.570724</td>
</tr>
<tr>
<td>0.002</td>
<td>993.12</td>
<td>970.7647</td>
<td>2.251017</td>
</tr>
<tr>
<td>0.001</td>
<td>501.14</td>
<td>473.5759</td>
<td>5.500279</td>
</tr>
</tbody>
</table>

After, a transversely isotropic beam made of E-glass epoxy was analyzed, the beam length is 0.5 m and width is 0.04 m. The properties used were: $E_x = 41$ GPa, $E_y = 10.4$ GPa, $\nu_{12} = 0.28$, $G_{12} = 9$ GPa and $\rho = 1970$ Kg/m$^3$. This led to a % difference between SOLIDWORKS and MATLAB as seen in Table 10. The difference was significant indicating the inadequacy of the approach.

Table 10: First twist frequency of E-glass epoxy beam with rectangular CS and sweeping height

<table>
<thead>
<tr>
<th>h/b</th>
<th>SOLIDWORKS [rad/sec]</th>
<th>MATLAB [rad/sec]</th>
<th>% diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>280.05</td>
<td>396.955</td>
<td>41.74433</td>
</tr>
<tr>
<td>0.02</td>
<td>187.05</td>
<td>260.95</td>
<td>39.50815</td>
</tr>
<tr>
<td>0.01</td>
<td>93.761</td>
<td>136.76</td>
<td>45.86022</td>
</tr>
</tbody>
</table>
3.5 - Discussion

Use of periodic analysis was shown to work very well for finding stop bands in isotropic and composite periodic beams. This work went through how stop bands are affected by different material and geometric parameters. After this, an attempt at making a 1D twist element for rectangular cross-sectional shafts was done. Sweeping of different parameters was performed in several studies. It was found that NC has one of the biggest effects in shifting the stop band and varying its size. When it comes to looking further into composite analysis, the angle of the ply was also a large change in the size of the stop band. When it comes to a 1D twist element, use of isotropic material produced results close to the analytical solution but when changing the length and material, the correction factor did not work as well. A more advanced beam model would be required to include twisting modes in the analysis.
Chapter 4 - Experimental Work

In order to validate the preceding theoretical modeling efforts, equivalent experiments were performed. Manufacturing of the beams and vibrational testing were done. This includes layup, material processing, and testing of the beams.

4.1 - Layup

In order to closely match the parameters used in computational modeling, a few hurdles had to be overcome. The main problem with laying up a composite beam was the need for the bottom surface to be flat against the mounting glass. There were three initial layup designs that were thought of. The first design was thought of to create the most closely related samples to the models created in SOLIDWORKS and in the computational code. The beam was to be all laid-up as one piece with the patches attached on top and bottom at once, as shown in Figure 59. The problem with this layup choice is the bending of the composite due to the bagging material pressing into the midsections.

![Composite Structure](image)

*Figure 59: Layup of composite structure as a single unit*

The second design that was attempted was one in which the main beam was placed first then periodic patches afterward as shown in Figure 60. This causes an issue with any non-symmetric layup composite structures. If any plies in the periodic section are anything other than 0 degrees, warping develops during the curing process due to different thermal coefficients in the different directions.
Finally, the selected beam design was a beam with periodic patches placed after curing. This was created first with a large plate that was then divided into periodic patches. The cutting and post processing of the patches allows for more accurate pieces to be made.

4.2 - Material Processing

After the layup and curing was completed, the creation of the periodic beams had to be done. For this, few steps had to be taken including, composite post processing, cutting Aluminum beams, adhering beams together and preparing completed beam for sensors to be bonded.

Taking the composite plates and Aluminum, then cutting into beams was done first. This was completed using a rotary tool and its cutting disc to cut to rough dimensions. Once the beam was cut, a final sanding could be done to get it closer to computational dimensions. Adhering the composite layers to the aluminum beam could then be done. Bonding the periodic patches to the host substrate beam was done using cyanoacrylate, or more commonly known as superglue. Before bonding was done, both the composite and Aluminum beams were sanded with 220 grit sandpaper, then cleaned with alcohol. The superglue was then applied to the composite beam, as seen in Figure 61. The beam was then placed in a position similar to the computational code. To make sure the glue is spread everywhere, a weight was placed on top of the composite beam.

Figure 60: Layup of composite structure without lower periodic patches

Figure 61: Application of cyanoacrylate to adhere composite periodic section to Aluminum beam
Once one side of the composite was done, the same process could be done to the other side. This finished the mating of the composite patches to the main beam. The last thing to do was to drill a hole in the beam in a location to mount the force sensor, seen in Figure 62. The location of the hole was chosen to be closer to the clamp side of the beam while also not being directly on a node as to not interfere with modal frequencies related to the first natural frequencies. Finally, as seen in Figure 63, the periodic beam was completed and ready for experiments.

Figure 62: Hole drilled for force sensor mounting

Figure 63: Completed periodic beam

4.3 - Setup

A completed setup of the cantilever configuration is shown in Figure 64. Figure 65 shows the equipment used to conduct the shaker vibration testing. The numbered items are major equipment without the cabling, beam and connectors. Table 11 shows the equipment shown in the figure and the others that are not pictured.
Figure 64: Completed shaker setup

Figure 65: Needed Equipment, minus cabling and beam, for shaker testing
Table 11: List of equipment for modal shaker testing

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Clamp</td>
</tr>
<tr>
<td>2</td>
<td>Laptop</td>
</tr>
<tr>
<td>3</td>
<td>Shaker</td>
</tr>
<tr>
<td>4</td>
<td>Clamps, hold downs</td>
</tr>
<tr>
<td>5</td>
<td>Data Physics Quattro SignalCalc Ace</td>
</tr>
<tr>
<td>6</td>
<td>mini ceramic shear accelerometer</td>
</tr>
<tr>
<td>7</td>
<td>ICP force sensor</td>
</tr>
<tr>
<td>8</td>
<td>Petro wax</td>
</tr>
<tr>
<td>9</td>
<td>Nylon Stringer</td>
</tr>
<tr>
<td>10</td>
<td>Coaxial Cables x2</td>
</tr>
<tr>
<td>11</td>
<td>Connector BNC to micro plug</td>
</tr>
</tbody>
</table>

To install the beam on to the clamp is a relatively simple process. Loosening two bolts on the side that holds down a bar across the clamp, slide the beam between the clamp and the bar, then tighten the beam down to the desired height and angle wanted, the finished product looks like that in Figure 66. Next is attaching the force sensor to the beam. There are multiple ways to adhere the sensor, the one chosen was drilling a hole into the beam. To make sure the force sensor was held on in a permanent way but also able to be removed easily, a bolted down connection was
chosen. The force sensor was tightened in a fashion to have its connector output be on the right, as to tighten when vibrating, or straight down.

![Image 1](image1.png)

*Figure 67: Showing location of accelerometer and tape to hold down wire*

After the force sensor was in place, the accelerometer was attached as shown in Figure 67. The sensor was adhered to the beam with petrol wax. The position was set as a location like the FEA, at the tip of the cantilever beam. Then in accordance to the shear accelerometers PDS, the wire was connected to the DAQ. Once, these two sensors have been adhered to the beam, external connections can be made.

![Image 2](image2.png)

*Figure 68: Aligning Stringer to force sensor*

External connections include the shaker to the force sensor, hold down clamps and wires. As seen in Figure 68, the shaker has a stringer that directs the output excitation to the force sensor, this stringer must be aligned with the sensor to make sure out-of-plane forces are limited. After this has been aligned, the clamps are placed to hold down the shaker and clamp to the table. Finally, all wires can be connected to the DAQ and experimental testing can be conducted.
4.4 - Vibration Analysis Verification

When utilizing the vibrational equipment, a verification that the measurements follow the numerical work or exact solutions. Two verifications were done to check the code before going further with tests. One test is with a completely isotropic beam.

4.4.1 - Isotropic Beam Testing

As seen in Table 12, when conducting the experimental modal analysis for an Aluminum beam, numerical and experimental results match within 10%.

Table 12: Comparison between numerical and experimental bending frequencies 17.75" × 1.71" × 0.09" 6061 Aluminum beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Numerical [Hz]</th>
<th>Experimental [Hz]</th>
<th>%Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.8</td>
<td>8.09</td>
<td>8.8</td>
</tr>
<tr>
<td>2</td>
<td>55.4</td>
<td>54</td>
<td>2.6</td>
</tr>
<tr>
<td>3</td>
<td>155.6</td>
<td>142</td>
<td>9.6</td>
</tr>
<tr>
<td>4</td>
<td>310.8</td>
<td>289</td>
<td>7.5</td>
</tr>
<tr>
<td>5</td>
<td>522.7</td>
<td>502</td>
<td>4.1</td>
</tr>
</tbody>
</table>

4.5 - Results

Before experimental tests were conducted, a numerical analysis had to be conducted on a beam like that which has to be used. All the gained knowledge of the effects of the different variables was utilized to be able to shift the stop band to a location near the first few natural frequencies and large enough to be perceived visually. This plot can be seen in Figure 69. The beam dimensions were: total length of 0.5334 m, width of 2.54 cm, 6061 Aluminum main beam, 1.587 mm thick, 3 cells, PSR= 0.7, 4 plies on each side for the periodic patches, 2 mm thick layers. Once a stop band was located, the beam in question could be created and then tested.
Figure 69: Computational frequency analysis on periodic and non-periodic composite beams

Figure 70 shows the results from vibration testing. There is an apparent gap in two frequencies between the third and fourth modes. This is in accordance to the computational code.

Figure 70: Experimental frequency analysis on periodic and non-periodic composite beams, highlighting area of stop bands

The removal of natural frequencies in the periodic beam is visible in the experimental results. Since the first frequency is so low, it does not visibly show, but a gap after the third mode that skips two frequencies is trending in the experimental work. This shows a good match between the computational and experimental results.
4.6 - Discussion

Using a physical test for verification of numerical results is always a necessary step in design. Testing done in this work was to create a beam that was similar to the beam being simulated. The trend of the experimental and the numerical results looks similar. However, more testing would need to be done before solid conclusions can be drawn about periodic analysis of composite beams with periodic patches attached. As seen in Figure 70, a stop band can be easily identified in the experimental frequency analysis. There seem to be another stop band next to the first, but this does not coincide with numerical results. There is also a difference in natural frequencies and location of the stop band. This could be due to different reasons. The material properties for the composite being tested were not completely known. This would lead to different estimates of natural frequencies. The material properties used for the numerical analysis were using a generic carbon-fiber reinforced material. Another issue could be the adhering of the patches to the beam, which could be incomplete and causes delamination in the layup, forcing waves to propagate in a fashion unlike the model used. Finally, some dimensions of the beam might not be precise during post processing. For a better understanding of the frequency response plots in relation to the computational code, a measuring device to sense displacements and not accelerations would be better. This would allow for amplitudes to also be correlated. Once more beams can be created and angle of the plies are varied in different samples, the effect should be more apparent in the data.
Chapter 5 - Conclusion

Numerically and experimentally analyzing stop bands in laminated fiber-reinforced composite beams was conducted. With the use of effective stiffness formulas, symmetric laminated composite beams can be analyzed. This research showed that with the use of composite materials, further variations of stop bands can be found. A more extensive research shows that there are more ways to design periodic beams for attenuation of specified frequencies. Composites give another layer on top of the already known characteristics of periodic beams with rectangular cross sections undergoing flexural vibrations. In this work, it is seen that natural frequencies are reduced as number of plies increases. SOLIDWORKS does not give the option to create frequency response plots for the beam in question, making investigating stop bands an impossible task. Thus, an in-house numerical code was created.

Looking at the periodic analysis of beams, the use of the dynamic stiffness matrix and transfer matrix was an amazing tool to be utilized because only one cell is analyzed. This work demonstrated the effects of all model parameters on the first two stop bands of periodic composite beams. For a non-symmetric beam such as a [0/90/90] layup or only having periodic patches on one side of the beam a more robust code would need to be used. This code would need to utilize a first-order or higher-order shear deformation theory. Similarly, the results in this work do not incorporate any bend-twist coupling due to material coupling or excitation sources. As stated in the literature review, there have been studies into these subjects. I do believe the use of symmetric beams is much more useful, since it is not often that one would want a material coupling of bend-twist during curing or in-service manipulation.

This research shows an availability to not only make the stop band larger but also give a way to shift the bandgap to lower or higher frequencies. Since energy is a function of frequency, it is commonly harder to excite higher modes.

Experimental modal analysis is a very powerful tool. Comparison of experiments and numerical solutions are needed to check if what was computed also works in the real world. In the experiments in this work, a limited amount of data was collected thus, solid evidence of whether the numerical results were a good representation of reality is inconclusive. But the data that was collected did give great preliminary results, showing an attenuation of frequencies in a similar region as numerical results. There were a lot of factors that could account for the differences, such
as material properties, numerical methods, processing of materials or unseen issues. Unseen issues include delamination or degradation of the composite patches adhered to the aluminum beam.

Twist of a beam that has a non-circular cross section is largely uninvestigated. The natural frequencies of a rectangular beam are nonlinear functions of beam thickness. This is due to the nonuniform shear stress across the surface of the beam.
References


[27] L. Rayleight, "On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure.." Phylosophical Magazine, pp. 145-159, 1887.


[60] "SignalCalc Ace," Data Physics, [Online]. Available: