

**STABILITY AND EXACT MULTIPLICITY OF  
PERIODIC SOLUTIONS OF DUFFING EQUATIONS  
WITH CUBIC NONLINEARITIES**

HONGBIN CHEN AND YI LI

ABSTRACT. We study the stability and exact multiplicity of periodic solutions of the Duffing equation with cubic nonlinearities,

$$x'' + cx' + ax - x^3 = h(t), \quad (*)$$

where  $a$  and  $c > 0$  are positive constants and  $h(t)$  is a positive  $T$ -periodic function. We obtain sharp bounds for  $h$  such that  $(*)$  has exactly three ordered  $T$ -periodic solutions. Moreover, when  $h$  is within these bounds, one of the three solutions is negative, while the other two are positive. The middle solution is asymptotically stable, and the remaining two are unstable.

1. INTRODUCTION

Consider the Duffing equation

$$(1.1) \quad x'' + cx' + ax - x^3 = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T),$$

where  $h$  is a positive  $T$ -periodic function and  $a$  and  $c$  are constants with  $0 < a < (\frac{\pi}{T})^2 + \frac{c^2}{4}$  and  $c > 0$ . The reason that we assume  $0 < a < (\frac{\pi}{T})^2 + \frac{c^2}{4}$  is that for  $0 > a$ , (1.1) has a unique  $T$ -periodic solution: there is no bifurcation at all. Therefore the only interesting case is  $a > 0$ . We assume that

$$(1.2) \quad a < (\frac{\pi}{T})^2 + \frac{c^2}{4}$$

Namely, the system is sufficient damped probable due to the drawback of our proof, we still don't know whether the condition (1.2) is essential or not. The existence and multiplicity of periodic solutions of (1.1) or more general types of nonlinear second-order differential equation have been investigated extensively by many authors. However, relatively few studies have been written about the exact multiplicity of (1.1). In [5], we have studied the small-perturbation problem, and established that (1.1) has exactly three  $T$ -periodic solutions provided that  $h(t)$  is sufficiently small. R. Ortega has considered the following parametrized Duffing equation[16]:

$$(1.3) \quad x'' + cx' + g(t, x) = s + h(t),$$

where  $c > 0$  and  $g'_x(t, x)$  is strictly increasing in the second variable. Under the additional assumptions

$$g'_x(t, x) \ll \frac{\pi^2}{T^2} + \frac{c^2}{4}$$

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and

$$\lim_{x \rightarrow \pm\infty} g(t, x) = +\infty,$$

he has shown the following Ambrosetti-Prodi-type theorem:

There is an  $s_0$  such that (1.3) has no  $T$ -periodic solution for  $s < s_0$ , (1.3) has a unique  $T$ -periodic solution which is unstable for  $s = s_0$ , and (1.3) has exactly two periodic solutions for  $s > s_0$ , one of them is asymptotically stable and another is unstable.

For related results on the existence of two periodic solutions, we refer to [7, 18, 9]. Similar results concerning the first-order equation were obtained by J. Mawhin [13], and more recently by the authors [3] based on singularity theory and A. Tineo [19]. For the multiplicity results concerning the forced pendulum equation, one can refer to [8, 10, 17]. However, as far as the authors know, there is no such precise result on existence of exactly three periodic solutions for nonlinear Duffing equations. Here, we obtain multiplicity for periodic solutions by means of a topological degree argument combined with a newly developed maximum principle given in [20, 15, 14], and stability of periodic solutions follows by computing the local index given by R. Ortega in [16]. The more recent results concerning the stability and the sharpness of rate of decay of periodic solutions can be found in [1, 2, 4, 11, 12].

**Theorem 1.1.** *Let  $h_0 := \sqrt{4a^3/27}$ . Then*

- (1) (1.1) has a unique  $T$ -periodic solution which is negative and unstable if  $h(t) > h_0 \forall t \in \mathbb{R}$ ;
- (2) (1.1) has exactly three ordered  $T$ -periodic solutions if  $0 < h(t) < h_0$ .
- (3) Moreover, in case (2), the minimal solution is negative and the other two are positive; also, the middle solution is asymptotically stable and the remaining two are unstable.

The following notations are used.

- (1)  $L_T^p$   $T$ -periodic function  $u \in L^p[0, T]$  with  $\|u\|_p$  for  $1 \leq p \leq \infty$ ;
- (2)  $C_T^k$   $T$ -periodic function  $u \in C^k[0, T]$ ,  $k \geq 0$ , with  $C^k$ -norm;
- (3)  $\alpha(t) \gg \beta(t)$ , if  $\alpha(t) \geq \beta(t)$  and  $\alpha(t) > \beta(t)$  on some positive-measure subset.

## 2. TOPOLOGICAL INDEX AND LINEAR PERIODIC PROBLEMS

In this section we shall recall some basic results about linear periodic boundary-value problems that will be needed in the sequel.

Consider the periodic boundary-value problem

$$(2.1) \quad \begin{cases} x' = F(t, x), \\ x(0) = x(T), \end{cases}$$

where  $F: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function that is  $T$ -periodic in  $t$  for  $n = 2$ .

We denote by  $x(t, x_0)$  the initial-value solution of (2.1).

**Definition.** *A  $T$ -periodic solution  $x$  of (2.1) will be called a nondegenerate  $T$ -periodic solution if the linearized equation*

$$(2.2) \quad y' = F_x(t, x)y$$

*does not admit a nontrivial  $T$ -periodic solution.*

Let  $M(t)$  be the fundamental matrix of (2.2), and  $\mu_1$  and  $\mu_2$  the eigenvalues of the matrix  $M(T)$ . Then  $x(t, x_0)$  is asymptotically stable if and only if  $|\mu_i| < 1$ ,  $i = 1, 2$ ; otherwise, if one of them has modulus greater than one,  $x(t, x_0)$  is unstable.

Consider the homogeneous periodic equation

$$(2.3) \quad L_\alpha x = x'' + cx' + \alpha(t)x = 0$$

where  $c$  is a constant and  $\alpha(t) \in L_T$ .

The following simple lemma is given by the authors in [5] which is needed in proving our main results.

**Lemma 2.1.** *Suppose that  $\alpha(t)$ ,  $\alpha_1(t)$  and  $\alpha_2(t) \in L_T$  such that*

$$(2.4) \quad \alpha_1(t), \alpha_2(t) \text{ and } \alpha(t) \ll \left(\frac{2\pi}{T}\right)^2 + \frac{c^2}{4}.$$

Then

- (1) *the possible  $T$ -periodic solution  $x$  of equation (2.3) is either trivial or different from zero for each  $t \in \mathbb{R}$ ;*
- (2)  *$L_{\alpha_i}x = 0$  ( $i = 1, 2$ ) cannot admit nontrivial  $T$ -periodic solutions simultaneously if  $\alpha_1(t) \ll \alpha_2(t)$ ;*
- (3)  *$L_\alpha x = 0$  has no nontrivial  $T$ -periodic solution, and  $\text{ind } L_\alpha := \text{deg}(L_\alpha) = 1$  (resp.,  $= -1$ ), if  $\alpha(t) \gg 0$  (resp.,  $\alpha(t) \ll 0$ ).*

The following connection between stability and topological index is due to R. Ortega [16].

**Lemma 2.2.** *Assume that  $x$  is an isolated  $T$ -periodic solution of (1.2) such that the condition*

$$g_x(t, x) \leq \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}$$

*holds, for  $t \in \mathbb{R}$ ,  $c > 0$ . Then  $x$  is asymptotically stable (resp., unstable) if and only if  $\text{ind}(x) = 1$  (resp.,  $\text{ind}(x) = -1$ ).*

Consider the differential equation

$$(2.5) \quad x'' + cx' + \alpha(t)x = h(t),$$

where  $c$  is a constant and  $\alpha(t)$ ,  $h(t) \in L_T$ .

The following maximum principle is given by P.J. Torres and M.R. Zhang (Theorem 2.3) in [20]. Here we state the principle in a somewhat different form.

**Lemma 2.3.** *Let  $h(t) \gg 0$  and  $\alpha(t)$  satisfy*

$$\alpha(t) \leq \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}.$$

*If  $x(t)$  is a  $T$ -periodic solution of (2.5), then the following statements hold:*

- (1) *Either  $x(t) > 0$  or  $x(t) < 0$  for all  $t \in \mathbb{R}$ ;*
- (2)  *$x(t) > 0 \forall t \in \mathbb{R}$  if  $\alpha(t) > 0 \forall t \in \mathbb{R}$ ;*
- (3)  *$x(t) < 0 \forall t \in \mathbb{R}$  if  $\alpha(t) < 0$ .*

*Proof.* If  $x(t)$  changes sign on  $[0, T]$ , then there is a  $\tau$  such that  $x(\tau) = 0$ . We may assume that  $x'(\tau) \leq 0$ . Otherwise, if  $x'(\tau) > 0$ , since  $x(t)$  is a  $T$ -periodic function,  $x(t)$  has a successive zero  $t_0$  on  $[\tau, \tau + T)$ , such that  $x'(t_0) \leq 0$ . In this case, let  $\tau = t_0$ . Without loss the generality, we may assume that  $\tau = 0$  so that  $x'(0) \leq 0$ . Let  $v(t)$  be initial value problem of the following equation

$$(2.6) \quad y'' - cy' + \alpha(t)y = 0,$$

such that  $v(0) = v'(0) - 1 = 0$ . by assumption that  $\alpha(t) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4} = \lambda_1$  the first eigenvalue of

$$y'' - cy' + \lambda x = 0, \quad x(0) = x(T) = 0.$$

Therefore the equation (2.6) disconjugate on  $[0, T]$ . Thus  $v(t) > 0 \forall t \in (0, T]$ . Multiplying (2.5) by  $v(t)$  subtracted from (2.6) by  $x(t)$ , and integration by parts, we have that

$$(2.7) \quad v(T)x'(T) = vx' - xv'|_0^T = \int_0^T v(t)h(t)dt.$$

The left side of above equation is negative, while the right side is positive, a contradiction.  $\square$

### 3. PROOF OF THEOREM 1.1

Before giving the proof of Theorem 1.1, let us list some results that will be needed in the sequel.

Consider the following Duffing equation:

$$(3.1) \quad x'' + cx' + g(t, x) = h(t),$$

where  $g(t, x)$  is  $T$ -periodic in  $t$  and differentiable in  $x$ .

**Lemma 3.1.** *Assume that*

$$(3.2) \quad g'_x(t, x) \ll \left(\frac{2\pi}{T}\right)^2 + \frac{c^2}{4}.$$

- (1) *The  $T$ -periodic solutions of (3.1) are totally ordered;*
- (2) *(3.1) has a unique  $T$ -periodic solution on  $[u, v]$  if a  $T$ -periodic solution exists and  $g'_x(t, x) \geq 0$  on  $[u, v]$ ;*
- (3) *(3.1) cannot admit three distinct  $T$ -periodic solutions in  $[u, v]$  if  $g'_x(t, x)$  is strictly increasing or strictly decreasing in  $[u, v]$ .*

*Proof.* The idea we use to prove the lemma is well known; we give the proof here for completeness.

Let  $x_1(t)$  and  $x_2(t)$  be two distinct  $T$ -periodic solutions of (3.1), and  $u = x_2(t) - x_1(t)$  a nontrivial  $T$ -periodic solution of (2.3) with  $\alpha(t) = \int_0^1 g'(t, (1-s)x_1 + sx_2) ds$ . Hence, the conclusion of (1) follows from Lemma 2.1(1). Similarly, (2) follows from Lemma 2.1(3). Next, let  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  be three distinct  $T$ -periodic solutions of (3.1) in the interval. By (1), we may assume that  $x_1(t) < x_2(t) < x_3(t)$ . Setting  $u_i = x_{i+1}(t) - x_i(t)$  for  $i = 1, 2$ , then  $L_{\alpha_i}(u_i) = 0$  with  $\alpha_i = [g(t, x_{i+1}) - g(t, x_i)]/u_i$ . The strict convexity of  $g$  implies that  $\alpha_1 < \alpha_2$ . By Lemma 2.1(2), we have  $u_1(t) \equiv 0$  or  $u_2(t) \equiv 0$ , a contradiction in either case.  $\square$

Now we turn to (1.1). Let  $Fx := x'' + cx' + ax - x^3 - h(t)$  and let  $B_R$  be a ball of radius  $R$ . We have the following lemma.

**Lemma 3.2.** *For any positive number  $a > 0$ , there is an  $R > 0$  large enough that  $\deg(F, B_R, 0) = -1$ .*

*Proof.* If  $x(t)$  is a  $T$ -periodic solution of (1.1), let  $t_{\max}, t_{\min}$  be points at which  $x(t)$  achieves its maximum and minimum, respectively. Then  $x'(t_{\max}) = 0$ ,  $x''(t_{\max}) \leq 0$ , hence (1.1) implies

$$ax(t_{\max}) - (x(t_{\max}))^3 \geq h(t_{\max}) > 0$$

so that

$$(3.3) \quad \max_t x(t) = x(t_{\max}) \in (-\infty, -\sqrt{a}) \cup (0, \sqrt{a}).$$

Similarly

$$ax(t_{\min}) - (x(t_{\min}))^3 \leq h(t_{\min}) < \|h\|_\infty$$

so

$$\min_t x(t) \geq C$$

where  $C$  is the negative root of  $g(C) = \|h\|_\infty$ . Thus any  $T$ -periodic solutions is bounded by

$$C \leq x(t) \leq \sqrt{a},$$

so the required a priori bound is proved. The fact that the degree is  $-1$  can also be seen by using the homotopy

$$(3.4) \quad x'' + cx' + ax - x^3 = \lambda h(t), \quad 0 \leq \lambda \leq 1.$$

For  $\lambda = 0$  is easy to compute directly that the degree is  $-1$ , and since the priori bound above holds for all  $\lambda \in [0, 1]$ , we conclude that the degree is  $-1$ .  $\square$

Now we are ready to prove Theorem 1.1.

First, we shall show that (1.1) has a unique  $T$ -periodic solution which is negative and unstable for  $h(t) > h_0 \forall t \in \mathbb{R}$ . It is obvious that  $-R$  is a constant subsolution of (1.1) for  $R$  large enough. By the choice of  $h_0$ , it is easy to verify that  $g(-2\sqrt{a/3}) = h_0 < h(t)$ , so  $b = -2\sqrt{a/3}$  is a supersolution of (1.1). By applying Lemma 3.2 in [21], there is a  $T$ -periodic solution  $x(t)$  of (1.1) such that

$$-R < x(t) < b.$$

Next, we have to show that the solution obtained above is the only  $T$ -periodic solution of (1.1). Suppose there is another  $T$ -periodic solution  $y(t)$  of (1.1): then  $y(t) < b := -2\sqrt{a/3}$ . In fact, let  $t_{\max}$  be the point at which  $x(t)$  achieves its maximum, then

$$(3.5) \quad ay(t_{\max}) - (y(t_{\max}))^3 \geq h(t_{\max}) > h_0,$$

which forces that  $\max_t y(t) < b$ , since  $g(y) \leq h_0$  when  $y > b$ .

Now that both  $x(t)$  and  $y(t)$  are in  $[-\infty, b]$ ,  $g$  is decreasing on this interval, so it follows from the second conclusion of Lemma 3.1 that  $x(t) \equiv y(t)$ . Finally, since the solution is unique, the local index of the  $T$ -periodic solution is equal to the degree given by Lemma 3.2, so the unique  $T$ -periodic solution is unstable.

This completes the proof of the first part of the theorem.

The idea of the proof of the second part is the same as that of the first one, but here we have to estimate the solutions more carefully.

First, we note that (1.1) does not admit any sign-changing  $T$ -periodic solutions. This follows directly by rewriting (1.1) as the following form:

$$(3.6) \quad x'' + cx' + q(t)x = h(t),$$

where  $q(t) = g(x(t))/x(t)$  verifies the condition of Lemma 2.3. Therefore,  $x(t)$  is of constant sign.

Next, we shall show the existence and uniqueness of a negative solution.

The existence is evident, since  $-\sqrt{a}$  and  $b$  are constant sub- and supersolutions of (1.1) respectively in the case  $0 < h(t) < h_0$ , so there is a  $T$ -periodic solution  $x_1(t)$  of (1.1) such that  $b < x_1(t) < -\sqrt{a}$ .

Letting  $x(t)$  be any negative solution of (1.1), similar argument as the proof of Lemma 3.2. from (3.3) we have

$$\max_t x(t) < -\sqrt{a}.$$

and from condition  $0 < h(t) < h_0$ , we have

$$ax(t_{\min}) - (x(t_{\min}))^3 \leq h(t_{\min}) < h_0$$

hence  $\min x(t) > b$  since  $g(x) \geq h_0$  for  $x \leq b$ . Therefore both  $x_1(t)$  and  $x(t)$  lie in the same interval  $(b, -\sqrt{a})$ . Moreover,  $g$  is decreasing on the interval, so Lemma 3.1 implies that  $x_1(t) \equiv x(t)$ . This establishes the uniqueness of the negative solution and shows that  $\text{ind}(x_1(t)) = -1$ .

Following the same reasoning as in the first part, there is a unique  $T$ -periodic solution  $x_3(t)$  in  $(\sqrt{a/3}, \sqrt{a})$  with  $\text{ind}(x_3(t)) = -1$ . According to a formula that relates local index and topological degree, there is another positive  $T$ -periodic solution  $x_2(t)$ . Since  $g$  is concave on  $(0, \infty)$ , by the third conclusion of Lemma 3.1 it is clear that on  $(0, \infty)$  (1.1) cannot admit more than two solutions. We infer that the positive  $T$ -periodic solutions are exactly two and  $\text{ind}(x_2(t)) = 1$ . This completes the proof of the theorem.

We will finish the paper by proposing the following open questions in the hope that the reader may solve it.

- (1) The main theorem characterizes the situation for positive forcing functions  $h(t)$  whose graph does not cross the line  $y = h_0$ . The conjecture naturally arises that, for any forcing  $h(t)$ , or at least for any positive  $h(t)$ , there are at most three periodic solutions (and generically either one or three solutions).
- (2) Is the condition (1.2) a sharp one for validity of the upper bound on the number of solutions? In other words, assuming that

$$a > \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4},$$

is it always possible to find a forcing  $h(t)$  with  $h(t) > h_0$  such that (1.1) has more than one solution, and a forcing  $h(t)$  with  $0 < h(t) < h_0$  such that (1.1) has more than three solutions?

- (3) We guess that the Duffing operator  $Fx := x'' + cx' + ax - x^3$  is globally equivalent to the cusp mapping in the sense of Berger and Church [6] under the condition of Theorem 1.1. Up to now, we have not been able to give a proof of it in the strict sense. So we state our question more precisely below:

Let  $\Sigma = \{x \in C_T^2 \mid DF(x)v = 0, v \neq 0\}$  be the singular set of  $F$ . Then  $\Sigma$  consists of the fold and cusp and  $F(\Sigma)$  divides the space  $C_T$  into two open components  $A_1, A_3$ . Let  $C$  be the subset that consists of the cusp point. Then  $F(C)$  is a co-dimension-one submanifold of  $F(\Sigma)$  such that

- (a) (1.1) has a unique  $T$ -periodic solution which is unstable for  $h(t) \in A_1 \cup F(C)$ ;
- (b) (1.1) has exactly three ordered  $T$ -periodic solutions if  $h(t) \in A_3$ . Moreover, the middle one is asymptotically stable and the remaining two are unstable.
- (c) (1.1) has exactly two ordered  $T$ -periodic solutions for  $h(t) \in F(\Sigma)/F(C)$ . Both of them are unstable.

The main difficulty here is to verify that  $F: \Sigma \rightarrow F(\Sigma)$  is one to one.

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DEPARTMENT OF MATHEMATICS, XI'AN JIAOTONG UNIVERSITY, XI'AN P.R. CHINA  
E-mail address: hbchen@mail.xjtu.edu.cn

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IA 52242, USA AND  
DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN, CHINA