Primary decomposition of adjacent minors and tensors

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Certification of Approval

I certify that I have read Primary decomposition of adjacent minors and tensors by Aswin Rangasamy Venkatesan and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirement for the degree Masters of Arts at San Francisco State University.

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Abstract

Binomial edge ideals are very well studied in the literature. In this thesis, we extend these ideals to tensors and call them binomial tensor edge ideals. We do this by generalizing the notion of minors of matrices to tensors. We prove that they are radical and characterize their minimal primes. For a special class called adjacent tensor binomial ideals, we provide a counting formula for the number of minimal primes. Moreover, we restrict the notion of prime sequences and prime collections defined by two different sets of authors to study the minimal primes of a more general class of ideals, the $3 \times 3$ adjacent minors of $3 \times n$ matrices. We prove that they are equivalent. Finally, using the characterization of the minimal primes of $3 \times 3$ adjacent minor ideals of $3 \times n$ matrices we present an approach to compute the minimal primes of the $3 \times 3$ adjacent minors of $4 \times n$ matrices.
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Introduction

To motivate this thesis we start with a problem. Let $M$ and $N$ be two $m \times n$ matrices whose entries are from $\mathbb{Z}_+$. Diaconis, Eisenbud, and Sturmfels in [6] asked: for a fixed set of $m \times n$ matrices $\mathcal{B} = \{B_1, \cdots, B_k\}$ with integer entries, are $M$ and $N$ connected via $\mathcal{B}$? More precisely, can we add or subtract $B_i$’s from $M$ to reach $N$ under some conditions? An approach based on commutative algebra is presented in [6] to answer this question. Let $R = k[x_{ij} : i = 1, \cdots, m, j = 1, \cdots, n]$ be the polynomial ring with variables $x_{ij}$ and let $I_\mathcal{B}$ be the ideal generated by $\prod_{i,j} x_{ij}^{B_{ij}^+} - \prod_{i,j} x_{ij}^{B_{ij}^-}$ where $B \in \mathcal{B}$, and $B_{ij}^+, B_{ij}^-$ are the positive and negative components of $B$. Then $M$ and $N$ are connected via $\mathcal{B}$ if and only if \[\prod_{i,j} x_{ij}^{N_{ij}} - \prod_{i,j} x_{ij}^{M_{ij}} \in I_\mathcal{B},\] [6, Theorem 1.1].

In our case, $R$ is a Noetherian ring, meaning any ideal of $R$ can be written as a finite intersection of primary ideals. This is the primary decomposition of an ideal. This is extremely useful because the decomposition gives certain conditions on the entries of $M$ and $N$ for them to be connected. So to characterize the set of matrices that are connected with respect to $\mathcal{B}$ it is enough to compute the primary decomposition of $I_\mathcal{B}$.

However, in general, the problem of characterizing the primary decomposition of an ideal is hard. Thus one might focus on the minimal primes of the ideals. In fact, for radical ideals, the primary decomposition is given by the intersection of minimal primes. Luckily the ideals which we deal with in Chapters 2 and 3 are radical so it is enough to compute the minimal primes. These types of problems arise in the field of algebraic statistics [13, Chapter 8].
One special case is given by the ideal $I_{\text{adj}}(2)$ generated by the $2 \times 2$ adjacent minors of a generic $m \times n$ matrix. For any $m$ and $n$, the minimal primes of $I_{\text{adj}}(2)$ have been characterized in [11]. In this thesis, we take two different directions for generalizing these ideals. First, we extend the notion of 2-minors of matrices to 2-sub-tensor minors of tensors. Secondly, we explore the $3 \times 3$ adjacent minors of $m \times n$ matrices studied in [11] and [3].

As mentioned above, in Chapter 2, we extend the notion of $2 \times 2$ minor of a matrix to a 2-sub tensor minor of a tensor and study the adjacent tensor binomial ideals. These ideals are nice because a lot is known about binomial ideals. In particular, there is a class of binomial ideals called lattice basis ideals which has been studied in [7]. Hosten and Shapiro [10] characterized the minimal primes of these ideals. Luckily, the adjacent minor tensor ideals are also lattice basis ideals so we use the so-called basis matrix defined in [7] and the irreducible sub-matrix of these basis matrices studied in [10] to characterize the minimal primes of our ideal. Furthermore, we give a counting formula for the number of minimal primes for a given tensor.

In Chapter 3, we extend the work of Herzog et al. [9] on binomial edge ideals of a $2 \times n$ matrix to binomial tensor edge ideals. These binomial edge ideals naturally occur in the study of conditional independence ideals in algebraic statistics. Given any simple graph $G$ on $n$ vertices we define the binomial edge ideals to be the ideal generated by $2 \times 2$ minors given by columns $\{i, j\}$ of a generic $2 \times n$ matrix with indeterminant entries such that $\{i, j\}$ is an edge of $G$. Herzog et al. characterize the minimal primes and prove that these ideals are radical for any simple graph $G$. In Chapter 3 we define binomial tensor edge ideals for
simple graphs and prove that they are radical. We also characterize their minimal primes.

Finally, we explore the ideals generated by $3 \times 3$ minors. These have already been studied by Hosten and Sullivant [11] and Mohammadi et al. [3]. The minimal primes of the adjacent $m$-minors of an $m \times n$ matrix have been characterized in [11]. This work introduced prime sequences and showed that the prime sequences give rise to minimal primes. The work in [3] studies even more general determinantal ideals and characterize their minimal primes. This is achieved through a concept called prime collections which gives the minimal primes. In this thesis, we restrict just to the $3 \times 3$ minors of $3 \times n$ generic matrix and reduce the definition of prime sequence and prime collection to this case. Subsequently, we prove that there is a bijection between the set of prime collections and prime sequences. This implies that the minimal primes coming from the prime collections are the same as the ones coming from prime sequences. Even though a lot is known about these ideals it is still an open problem to characterize the minimal primes of the $3 \times 3$ adjacent minors of a $4 \times n$ matrix. In Chapter 4, we present an approach. We state a conjecture on a subset of the minimal primes of these ideals based on the work in [3].
Chapter 1

Background

Let us start by defining the most fundamental object called a ring.

**Definition 1** (Ring). A ring is a set $R$ with two binary operations $(+, \cdot)$ which satisfies these conditions:

- $R$ is an abelian group under $+$; we will denote $R$ with the binary operation $+$ as $(R, +)$.

- $R$ is associative under the operation $\cdot$; here associative means, for any $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. We will denote $R$ with the binary operation $\cdot$ as $(R, \cdot)$.

- $R$ has an identity element $1$ such that for any $r \in R$, $1 \cdot r = r \cdot 1 = r$.

- The operation $(\cdot)$ is distributive with the operation $(+)$; Distributive means, for any $a, b, c \in R$, $(a + b) \cdot c = a \cdot c + b \cdot c$. We will denote $R$ with the binary operations $+$ and $\cdot$ as $(R, +, \cdot)$.
A commutative ring $R$ is a ring where, $(R, \cdot)$ is commutative. In this work all the rings we deal with are commutative. Moreover, a ring $R$ with $(R \setminus \{0\}, \cdot)$ being a group is a field.

**Example 1.** The set of integers $\mathbb{Z}$ under the usual operation of addition (+) and multiplication (\cdot) is a commutative ring.

Given a ring, we are going to add more elements to it but will make sure that the whole set will again be a ring.

**Definition 2** (Polynomial ring). Let $R$ be a ring. The polynomial ring in one variable over $R$ is the set

$$R[x] = \{ r_0 + r_1 x^1 + r_2 x^2 + \cdots + r_n x^n \mid r_i \in R \text{ and } n \in \mathbb{N} \}.$$

The addition and multiplication in $R[x]$ are the familiar operations based on adding and multiplying polynomials with real coefficients, and thus $R[x]$ is also a ring. Note that we can also define polynomial rings in more than one variable.

**Example 2.** Let $R = \mathbb{Q}[x]$ be a polynomial ring over the field of rational numbers. An example of an element of this ring is $\frac{5}{2} x^3 + \frac{2}{3} x + 5$ but the element $x^2 + \pi x + \sqrt{2}$ is not in $R$ because $\pi, \sqrt{2} \notin \mathbb{Q}$.

**Definition 3** (Ideal of a Ring $R$). An ideal $I$ of $R$ is a subset of the ring $R$ satisfying the following conditions:

- $(I, +)$ is a subgroup of $(R, +)$. 
• For any \( a \in I \) and for any \( r \in R \), \( r \cdot a \in I \).

**Example 3.** 1. Let \( R = \mathbb{Z} \). Then the set of all even numbers \( I \), which is also denoted by \( 2\mathbb{Z} \), is an ideal of \( \mathbb{Z} \). If we add any two even integers we again get an even integer and this fact implies that \((I, +)\) is a subgroup of \((R, +)\). And similarly, if we multiply any integer with an even integer we get an even integer as a result. So the second condition is also satisfied.

2. Let \( R = \mathbb{Q}[x] \). Then the set \( I = \langle x \rangle = \{fx | f \in \mathbb{Q}[x]\} \) is an ideal of \( \mathbb{Q}[x] \). If \( f_1x, f_2x \in I \), then \( f_1x + f_2x = (f_1 + f_2)x \in I \). Since \( \mathbb{Q}[x] \) is an abelian group under +, \((I, +)\) will be a subgroup of \((R, +)\). And similarly let \( f_1 \in \mathbb{Q}[x] \) and \( g \in I \) then by definition, \( g = f_2x \) for some \( f_2 \in \mathbb{Q}[x] \), and \((f_1)(f_2x) = (f_1 \cdot f_2)x = f_3x \in I \).

More generally, we could define an ideal generated by a set of elements from the ring. Let \( R \) be a ring and let \( f_1, f_2, \ldots, f_n \in R \). Then the ideal generated by \( f_1, f_2, \ldots, f_n \) is

\[
I = \langle f_1, f_2 \cdots, f_n \rangle = \{r_1f_1 + r_2f_2 + \cdots + r_nf_n | r_i \in R\}.
\]

There is a special class of rings where any ideal of it has a finite generating set.

**Definition 4** (Noetherian ring). A ring \( R \) is a Noetherian ring if it satisfies the following three equivalent conditions:

• Any ideal \( I \) of \( R \) is generated by finitely many elements.

• Let \( \Omega \) be a non-empty set of ideals of \( R \). Then \( \Omega \) has a maximal element \( M \), i.e., \( M \) is not contained strictly in any of the ideals in \( \Omega \).
• Any ascending chain of ideals stabilizes, i.e., given an ascending chain of ideals

\[ I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \]

there exists an \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( I_n = I_N \).

Next, we will state one of the most celebrated theorems in the field of commutative algebra and algebraic geometry called Hilbert’s basis theorem. This was proved by David Hilbert in 1890.

**Theorem 1** (Hilbert’s Basis Theorem [4]). If \( R \) is a Noetherian ring then the polynomial ring \( R[x] \) is also Noetherian. Similarly, the polynomial ring \( R[x_1, x_2, \cdots, x_n] \) is Noetherian.

We defined what an ideal is, and now we define a quotient ring and some special classes of ideals.

**Definition 5** (Quotient ring). Let \( I \) be an ideal of a commutative ring \( R \), which is an abelian subgroup of \((R, +)\). Then \((R/I, +, \cdot)\) a ring structure where,

- \((R/I, +)\) is quotient group, and

- \(R/I\) has multiplication given by \((a + I) \cdot (b + I) = ab + I\) with the multiplicative identity \(1 + I\).

**Example 4.** We saw that \(2\mathbb{Z}\) is an ideal in \(\mathbb{Z}\). Then \(\mathbb{Z}/2\mathbb{Z} = \{1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}\}\), where \(1 + 2\mathbb{Z}\) is the multiplicative identity and \(0 + 2\mathbb{Z}\) is the additive identity.
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Definition 6 (Prime ideals). Let $I$ be an ideal of the ring $R$. Then $I$ is a prime ideal of $R$ if
\[ fg \in I \implies f \in I \text{ or } g \in I. \]

Example 5. 1. Let $p \in \mathbb{Z}$ be a prime number, and let $I = \langle p \rangle$. We will show that this ideal is a prime ideal. If $ab \in I$, then there is $c \in \mathbb{Z}$ such that $cp = ab$. So $p$ divides $ab$ and this implies that either $p$ divides $a$ or $b$. Hence either $a$ is in $I$ or $b$ is in $I$.

2. If $J = \langle 6 \rangle \subset \mathbb{Z}$ then $J$ not a prime ideal, since $2 \cdot 3 = 6 \in J$ but neither 2 nor 3 is in $J$. Thus $J$ is not a prime ideal.

Being a prime ideal is a strong requirement. But there is also another class of ideals which is less strong and will be very useful in our study.

Definition 7 (Primary ideals). Let $R$ be a ring. An ideal $I$ is a primary ideal of $R$ if $fg \in I$ implies $f \in I$ or $g^n \in I$ for some $n \in \mathbb{N}$.

Example 6. 1. Let us again take the ring of integers $\mathbb{Z}$. We will show that the ideal generated by powers of a prime number is primary. Let $n \in \mathbb{Z}$ and $p \in \mathbb{N}$ be a prime, and let $I = \langle p^n \rangle$. If $ab \in I$ then $ab = cp^n$ for some $c \in \mathbb{Z}$. So $p$ divides $ab$ and hence $p$ divides $a$ or $p$ divides $b$. Suppose $p$ does not divide $a$. This means no power of $a$ will have a factor of $p$ in it. Therefore $b = p^nc'$ for some $c' \in \mathbb{Z}$ and we conclude that $b \in I$.

Now suppose that $a$ does have a factor of $p$ but $a \notin I$. This means $p^m$ for some $m < n$ divides $a$ and $p^{n-m}$ should divide $b$. We see that $(p^n)^{(n-m)}$ divides $b^n$ and therefore $b^n \in I$. This shows that $I$ is primary.
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2. In contrast, if $J = \langle 12 \rangle$, then $3 \cdot 4 = 12 \in J$ but neither a power of 3 nor a power of 4 is in $J$. So $J$ is not primary.

**Definition 8** (Radical of an ideal $I$). Let $R$ be a ring and $I$ be an ideal of $R$. Then the radical of $I$, denoted by $\text{rad}(I)$, is an ideal which is defined to be

$$\text{rad}(I) = \{ f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N} \}.$$ 

An ideal $I$ is a radical ideal if $\text{rad}(I) = I$.

We see that by definition $I \subset \text{rad}(I)$.

**Theorem 2.** [12] The radical of a primary ideal is prime

**Proof.** Let $I \subset R$ be primary, and let $ab \in \text{rad}(I)$. Then $(ab)^n = a^n b^n \in I$ for some $n$. As $I$ is primary either $a^n \in I$ or $(b^n)^m \in I$ for some $m$. If $a^n \in I$ then $a \in \text{rad}(I)$, and if $(b^n)^m \in I$ then $b \in \text{rad}(I)$. Therefore either $a \in \text{rad}(I)$ or $b \in \text{rad}(I)$. Hence $\text{rad}(I)$ is prime. \hfill \Box

1.1 Primary Decomposition and Minimal Primes

**Primary Decomposition**

**Theorem 3.** [12] In a Noetherian ring $R$, every ideal $I$ is written as a finite intersection of primary ideals $Q_i$:

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_n.$$ 

For the proof of this theorem we need some definitions and lemmas.
Definition 9. An ideal $I$ of $R$ is **irreducible** if for two ideals $J, K$, $I = J \cap K$ implies $I = J$ or $I = K$. If $I$ is not irreducible then $I$ is **reducible**.

**Lemma 1.** [12] In a Noetherian ring $R$ any ideal $I \subset R$ is a finite intersection of irreducible ideals.

**Proof.** Suppose for a contradiction that there exists an ideal which cannot be written as a finite intersection of irreducible ideals. Then let $\Omega$ be the set of those ideals of $R$ which cannot be written as a finite intersection of irreducible ideals. By Definition 4, we can find a maximal element $M$ of $\Omega$. It is clear that $M$ is reducible because otherwise, it would be a finite intersection of irreducible ideals. Since $M$ is reducible, there exists ideals $J, K \neq M \subset R$ such that $M = J \cap K$. But since $M$ is a maximal element of $\Omega$, $J$ and $K$ are not in $\Omega$. Therefore $J$ and $K$ are finite intersections of irreducible ideals. But this is a contradiction because then $M$ can also be written as a finite intersection of irreducible ideals. So $\Omega$ is an empty set. \[\square\]

**Lemma 2.** Every proper irreducible ideal is a primary ideal.

**Proof.** By passing to the quotient ring $R/I$, it is enough to prove the statement for the zero ideal. Let $\langle 0 \rangle \subset R$ be the zero ideal. If we suppose $xy = 0$, then we should prove that either $x = 0$ or $y^n = 0$ for some $n \in \mathbb{N}$. It is easy to check that the set $Ann(y) = \{r \in R | ry = 0\}$ is an ideal. Consider the ascending chain of ideals

$$Ann(y) \subset Ann(y^2) \subset \cdots.$$
Here $\text{Ann}(y) \subset \text{Ann}(y^2)$, because

\[
a \in \text{Ann}(y) \implies ay = 0 \implies ay^2 = a \cdot y \cdot y = 0 \implies a \in \text{Ann}(y^2).
\]

In general, for all $n \in \mathbb{N}$, $\text{Ann}(y^n) \subset \text{Ann}(y^{n+1})$. Since $R$ is Noetherian, any ascending chain of ideals must stabilize. So there exists an $N \in \mathbb{N}$ such that for all $n \geq N \text{Ann}(y^N) = \text{Ann}(y^n)$.

We claim that $\langle x \rangle \cap \langle y^N \rangle = \langle 0 \rangle$. Then by the irreducibility of $\langle 0 \rangle$ we conclude that $\langle x \rangle = \langle 0 \rangle$ or $\langle y^N \rangle = \langle 0 \rangle$, thus showing that $x = 0$ or $y^N = 0$.

Let $a \in \langle x \rangle \cap \langle y^N \rangle$. Then $a = r_1x$ and $a = r_2y^N$ for some $r_1, r_2 \in R$. Since $\langle y^N \rangle = \langle y^{N+1} \rangle$, we have that $r_2y^{N+1} = r_2y^Ny = r_1xy = 0$, thus $r_2 \in \text{Ann}(y^{N+1}) = \text{Ann}(y^N)$, so $r_2y^N = 0 = a$. This shows that $a = 0$ and we conclude that $\langle x \rangle \cap \langle y^N \rangle = \langle 0 \rangle$. This proves the lemma.

\begin{proof}
Proof. Theorem 3. Since by Lemma 1 any ideal in a Noetherian ring is a finite intersection of irreducible ideals and by Lemma 2 any irreducible ideal is primary, this implies that any ideal in a Noetherian ring is a finite intersection of primary ideals.
\end{proof}

Minimal Primes

As we discussed in the previous section any ideal $I$ of a Noetherian ring can be written as a finite intersection of irreducible (primary) ideals. In general the primary decomposition of an ideal is not unique. In this section we are going to look at the radical of an ideal and prove that it has a unique primary decomposition.
**Definition 10.** Let $Q \subset R$ be a primary ideal. We say that $Q$ is $P$-primary for the prime ideal $P$ if $P = \text{rad}(Q)$.

**Definition 11 (Quotient ideal).** Let $I$ be an ideal of the ring $R$, and let $x \in R$. We define the quotient ideal of $I$ with respect to $x$ as

$$(I : x) = \{ r \in R \mid r \cdot x \in I \}.$$ 

**Lemma 3.** Let $Q \subset R$ be a $P$-primary ideal. Then, for any $x \in R$,

- $(Q : x) = R$ if $x \in Q$,
- $(Q : x)$ is $P$-primary if $x \notin Q$,
- $(Q : x) = Q$ if $x \notin P$. Furthermore, if $R$ is Noetherian, there exists an element $x \in R$ such that $(Q : x) = P$.

**Proof.**

- If $x \in Q$ then $(Q : x) = \{ r \in R \mid rx \in Q \}$ and $1 \in (Q : x)$ proving that $(Q : x) = R$.

- If $x \notin Q$ we need to prove that $\text{rad}(Q : x) = P$. Let $a \in \text{rad}(Q : x)$. Then we have that $a^n \in (Q : x)$ for some $n \in \mathbb{N}$. So $a^n x \in Q$. However, we know that $x \notin Q$ and therefore $(a^n)^m \in Q$ for some $m$. This implies that $a \in \text{rad}(Q)$, and hence $\text{rad}(Q : x) \subset \text{rad}(Q)$. The other inclusion is clear.

- Clearly $Q \subset (Q : x)$. Conversely, if $a \in (Q : x)$, that is $ax \in Q$. Suppose $a \notin Q$ then $x^n \in Q$ as $Q$ is primary. But this contradicts the fact $x \notin P$. So $a \in Q$. 


If \( R \) is Noetherian, then \( \text{rad}(Q) = P \) is finitely generated. This implies that \( P^n \subset Q \) for some \( n \in \mathbb{N} \). Let us pick an \( n \in \mathbb{N} \) such that \( P^n \subset Q \) but \( P^{n-1} \not\subset Q \). Then let \( x \in P^{n-1} \setminus Q \). Since \( x \notin Q \) we have that \( \text{rad}(Q : x) = \text{rad}(Q) = P \). So \( (Q : x) \subset P \). However, because \( x \cdot P \subset P^n \) and \( P^n \subset Q \) we have \( P \subset (Q : x) \). Both of these imply that \( P = (Q : x) \).

\[ \square \]

**Lemma 4.** The intersection of two \( P \)-primary ideals is \( P \)-primary.

**Proof.** Let the ideals \( Q_1, Q_2 \) be \( P \)-primary and let \( Q = Q_1 \cap Q_2 \). First we will show that \( Q \) is primary. Let \( xy \in Q \), suppose \( x \notin Q \) then either \( x \notin Q_1 \) or \( x \notin Q_2 \) or \( x \notin Q_1, Q_2 \).

Suppose \( x \notin Q_1 \), then \( y^n \in Q_1 \) which implies \( y \in \text{rad}(Q_1) = \text{rad}(Q_2) = P \). Which further implies \( y^m \in Q_2 \) and hence \( y^{mn} \in Q \). If \( x \notin Q_2 \) then similar argument works. Now suppose \( x \notin Q_1, Q_2 \), since \( xy \in Q_1, Q_2 \) implies \( y^n \in Q_1 \) and \( y^m \in Q^2 \) so \( y^{mn} \in Q \). Hence \( Q \) is primary.

To show \( \text{rad}(Q) = P \), first let us observe that \( \text{rad}(Q_1 \cap Q_2) = \text{rad}(Q_1) \cap \text{rad}(Q_2) \). Let \( x \in \text{rad}(Q_1 \cap Q_2) \), then \( x^n \in Q_1 \cap Q_2 \) for some \( n \). So \( x \in \text{rad}(Q_1) \) and \( x \in \text{rad}(Q_2) \) which implies \( x \in \text{rad}(Q_1) \cap \text{rad}(Q_2) \). To prove the other inclusion let \( x \in \text{rad}(Q_1) \cap \text{rad}(Q_2) \). Then \( x^n \in Q_1 \) and \( x^m \in Q_2 \) for some \( m, n \), which implies that \( x^{mn} \in Q_1 \cap Q_2 \). So \( x \in \text{rad}(Q_1 \cap Q_2) \). Therefore, \( \text{rad}(Q_1 \cap Q_2) = \text{rad}(Q_1) \cap \text{rad}(Q_2) = P \).

\[ \square \]

**Definition 12** (Irredundant primary decomposition). Let \( I \) be an ideal of a Noetherian ring \( R \). A primary decomposition \( I = \bigcap_{i=1}^{r} Q_i \) is called irredundant if
• for any \( j \), \( I \not\subseteq \cap_{i \neq j} Q_i \) and

• If \( Q_j \) is \( P \)-primary for some \( j \), then \( \forall i \neq j, Q_i \) is not \( P \)-primary.

**Theorem 4.** [12] Let \( I \) be an ideal of a Noetherian ring \( R \), and let

\[
I = \bigcap_{i=1}^{r} Q_i
\]

be an irredundant primary decomposition. Then the set of corresponding prime ideals \( P_i = \text{rad}(Q_i) \) is independent of the chosen decomposition and depends only on the ideal \( I \). More precisely, it coincides with the set of all prime ideals in \( R \) that are of type \( \text{rad}(I : x) \) for \( x \) varying over \( R \).

This theorem tells us that given an ideal \( I \), the primary decomposition may not be unique to \( I \). However, the set of prime ideals \( P_i = \text{rad}(Q_i) \) is independent of the chosen decomposition. So given \( I \) we can associate to it a set of prime ideals. Moreover, these prime ideals are of the form \( \text{rad}(I : x) \) for some \( x \in R \).

**Definition 13.** Let \( I \subset R \) be an ideal. The set of all prime ideals in \( R \) that are of type \( \text{rad}(I : x) \) for some \( x \in R \) is denoted by \( \text{Ass}(I) \). Its members are called the prime ideals associated to \( I \).

The following theorem is an extension of Theorem 4.

**Theorem 5.** [12] Let \( I \subset R \) be an ideal where \( R \) is a Noetherian ring. Then a prime ideal \( P \subset R \) belongs to \( \text{Ass}(I) \) if and only if it is of type \( (I : x) \) for some \( x \in R \). Moreover, \( \text{Ass}(I) \) is finite.
Definition 14 (Minimal primes and embedded primes). For any ideal $I \subset R$, the subset of all prime ideals in $Ass(I)$ that are minimal with respect to inclusion is denoted by $\text{minPrimes}(I)$, and its members are called the minimal primes of $I$. All other elements of $Ass(I)$ are said to be embedded primes of $I$.

Example 7. Let $I = \langle x^2, xy \rangle \subset \mathbb{Q}[x, y]$. Two of the irredundant primary decompositions of $I$ are

$$I = \langle x \rangle \cap \langle x^2, xy, y^2 \rangle = \langle x \rangle \cap \langle x^2, y \rangle.$$ 

The associated primes of $I$ are

$$Ass(I) = \{ \langle x \rangle, \langle x, y \rangle \}.$$ 

And since $\langle x \rangle \nsubset \langle x, y \rangle$, $minPrimes(I) = \{ \langle x \rangle \}$.

Lemma 5. [12] If $I$ is radical in $R$ then $Ass(I) = minPrimes(I)$.

Proof. Let $I = \bigcap_{i=1}^{r} Q_i$ be an irredundant primary decomposition. Then $rad(I) = \bigcap_{i=1}^{r} P_i$, where $P_i = rad(Q_i)$. If for some $i, j$, $P_i \subset P_j$ then $rad(I) = \bigcap_{i=1, i \neq j}^{r} P_i$. Since $\bigcap_{i=1}^{r} Q_i$ is an irredundant primary decomposition, $Q_i \nsubset Q_j$. So there is an $x \in \bigcap_{i \neq j} Q_i \setminus Q_j$. This implies $x \in \bigcap_{i=1, i \neq j}^{r} P_i = rad(I)$. Since $I$ is radical, $x \in I$ which implies, $x \in Q_j$, which is a contradiction. So $\forall i, j$, $P_i \nsubset P_j$. Hence $Ass(I) = minPrimes(I)$. $\square$
1.2 Grobner bases

One of the most useful tools in commutative algebra and algebraic geometry are Grobner bases. Given any finite set of polynomials we can compute a Grobner basis which has nice algorithmic properties. In the present day, commutative algebra and algebraic geometry softwares like Macaulay-2 [8], Singular [5], CoCoA [1] and many others use Grobner bases as their computational tool. In this section, we will start with some background and we will see what a Grobner basis is and how to compute it.

Throughout this section we will be using $R = k[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field $k$ in variables $x_1, x_2, \ldots, x_n$.

**Definition 15 (Monomials).** Let $R = k[x_1, x_2, \ldots, x_n]$, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ be a vector. Then any element of the form $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \in R$ is called a monomial.

**Definition 16 (Monomial order).** A monomial order on $k[x_1, x_2, \ldots, x_n]$ is a relation $>$ on the set of monomials in $k[x_1, x_2, \ldots, x_n]$ satisfying the following:

- $>$ is a total order.
- If $x^\alpha > x^\beta$, then for all $x^\gamma$, $x^\alpha \cdot x^\gamma > x^\beta \cdot x^\gamma$.
- $>$ is a well-ordering of the set of monomials: Every non-empty subset of the monomials in $k[x_1, x_2, \ldots, x_n]$ has a smallest element with respect to $>$.

**Example 8** (Lexicographical order). $x^\alpha >_{\text{lex}} x^\beta$ if the left most non-zero coordinate of $\alpha - \beta$ is positive. Let $k[x_1, x_2, x_3]$ be the polynomial ring on three variables, and let $\alpha = (3, 1, 1)$.
and \( \beta = (2, 3, 5) \) and \( \gamma = (3, 1, 2) \). Then by the definition \( x^\alpha = x_1^3x_2x_3^1 \), \( x^\beta = x_1^2x_2^3x_3^5 \) and \( x^\gamma = x_1^3x_2^1x_3^2 \). Since \( \alpha - \beta = (1, -2, -4) \) the left most non-zero entry is 1, this means that \( x_1^3x_2^1x_3^1 >_{\text{lex}} x_1^2x_2^3x_3^5 \). Similarly, \( \beta - \gamma = (-1, 2, 3) \) the left most non-zero entry is \(-1\), not positive, so \( x^\beta \) is not greater than \( x^\gamma \). Hence \( x_1^3x_2^1x_3^2 >_{\text{lex}} x_1^2x_2^3x_3^5 \).

Example 9 (Graded lexicographical order). \( x^\alpha >_{\text{grlex}} x^\beta \) if \( |\alpha| > |\beta| \) where \( |\alpha| = \sum_{i=1}^{n} \alpha_i \), or if otherwise \( |\alpha| = |\beta| \) and the left most non-zero entry of \( \alpha - \beta \) is positive. For example, let \( k[x_1, x_2, x_3] \) be the polynomial ring on three variables, and let \( \alpha = (3, 1, 1) \) and \( \beta = (2, 1, 3) \) and \( \gamma = (3, 1, 2) \). Since \( |\alpha| = 3 + 1 + 1 = 5 < |\beta| = 2 + 1 + 3 = 6 \), \( x^\beta >_{\text{grlex}} x^\alpha \). And \( |\beta| = 6 = |\gamma| = 3 + 1 + 2 = 6 \), and \( \gamma - \beta = (1, 0, -1) \), we get \( x^\gamma >_{\text{grlex}} x^\beta \).

Definition 17. Let \( R = k[x_1, \cdots, x_n] \) and \( f \neq 0 \in R \), and let \( > \) be a monomial order. Since \( f \) is a polynomial in \( R \), it can be written as a finite sum \( f = \sum c_\alpha x^\alpha \), where \( c_\alpha \in k \) and \( \alpha \in \mathbb{Z}_+^n \).

- Leading term \( LT_>(f) \): The leading term of \( f \) with respect to \( > \) is \( c_\beta x^\beta \) where \( x^\beta \) is the largest monomial with respect to \( > \) in the set \( \{ x^\alpha \mid c_\alpha \neq 0 \} \).
- Leading monomial \( LM_>(f) \): The leading monomial of \( f \) is \( x^\beta \) where \( LT_>(f) = c_\beta x^\beta \).
- Leading coefficient \( LC_>(f) \): The leading coefficient of \( f \) is \( c_\beta \) where \( LT_>(f) = c_\beta x^\beta \).
- Multidegree \( \text{multideg}(f) \): The multidegree of \( f \) is \( \beta \) where \( LT_>(f) = c_\beta x^\beta \).

Observation Let \( f, g \in R \) and let \( > \) be a monomial order on \( R \), then
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• $\text{LT}_>(fg) = \text{LT}_>(f)\text{LT}_>(g)$, and

• if $f + g \neq 0$ then $\text{multideg}(f + g) \leq \max\{\text{multideg}(f), \text{multideg}(g)\}$.

**Theorem 6.** [4, Division algorithm] Let $>$ be a monomial order on $R = k[x_1, x_2, \cdots, x_n]$ and $\{f_1, f_2, \cdots, f_s\} \subset R$ be an ordered set of polynomials, i.e $\text{LT}(f_1) > \cdots > \text{LT}(f_s)$. Then, given any polynomial $f$, there exist $q_1, q_2, \cdots, q_s, r \in R$ such that

$$f = q_1 f_1 + q_2 f_2 + \cdots + q_s f_s + r,$$

where either $r = 0$ or no term of $r$ is divisible by any $\text{LT}_>(f_i)$.

Even though we are not going to prove this theorem we will illustrate an example that should convince us that the division algorithm works.

**Example 10.** Let $R = \mathbb{Q}[x, y]$ under the $>_\text{lex}$ term order. Let $f = x^2y + xy^2 + y^2$ and let $f_1 = y^2 - 1$ and $f_2 = xy - 1$. Our goal is to divide $f$ by $f_1, f_2$. By our term order, we can see that $\text{LM}(f_1) < _\text{lex} \text{LM}(f_2)$. Let us start with the polynomial whose leading monomial is the largest which is $f_2$. We first try to cancel the leading monomial of $f$ with $f_2$. Because $\text{LM}(f) = x^2y$, if we multiply $x$ with $f_2$ we get $x^2y - x$ and then

$$f - xf_2 = xy^2 + x + y^2.$$

Next we divide the above polynomial by $f_1$. Since $x \cdot f_2 = xy^2 - x$ we have

$$f - xf_2 - xf_1 = xy^2 + x + y^2 - xy^2 + x = 2x + y^2.$$

Here the leading term is $2x$ which cannot be divided by any of the leading term of $f_1$ and $f_2$, so this should go to the reminder. However, we have $y^2$ as the next leading term which can be canceled by $f_1 \cdot 1$, so we get

$$2x + y^2 - y^2 + 1 = 2x + 1.$$ 

Here none of the terms is divisible by the leading terms of $f_1$ and $f_2$. We stop and let $r = 2x + 1$ and and $q_1 = x + 1$ and $q_2 = x$:

$$f = q_1 f_1 + q_2 f_2 + r.$$ 

**Definition 18** (Monomial Ideals). An ideal $I \subset R = k[x_1, x_2, \ldots, x_n]$ is a monomial ideal if there exists a subset $A \subseteq \mathbb{Z}^n_+$ such that $I = \langle x^\alpha : \alpha \in A \rangle$.

**Lemma 6.** [4] Let $I$ be a monomial ideal in $k[x_1, x_2, \ldots, x_n]$. Then

$$f \in I \iff \text{every term of } f \text{ is in } I.$$ 

**Proof.** One direction is trivial because if every term of $f$ is in $I$ then $f \in I$. To prove the other direction, observe that $f = h_1 x^{\alpha(1)} + h_2 x^{\alpha(2)} + \cdots + h_s x^{\alpha(s)}$ where $\alpha(i) \in A$ and $h_i \neq 0 \in R$. Then for all $i$ each term of $h_i$ is multiplied by the generators of the monomial ideal $I$. So each term of $f$ is in $I$. \qed

**Proposition 1.** [4] Let $I = \langle x^\alpha : \alpha \in A \rangle$ be a monomial ideal where $A \subseteq \mathbb{Z}^n_+$. Then there exists $s$ such that $I = \langle x^{\alpha(i)} : \alpha(i) \in A; i = 1, 2, \ldots, s \rangle$. 
Proof. Let us pick a term order $>$ and order the monomials from the set $A$ as

$$x^{\alpha(1)} < x^{\alpha(2)} < x^{\alpha(3)} \cdots.$$ 

Now let $I_j = \langle x^{\alpha(1)}, x^{\alpha(2)}, \cdots, x^{\alpha(j)} \rangle$. From this we can see that

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I.$$ 

As $R$ is a Noetherian ring the ascending chain of ideals above must stabilize. So there exists an $s \in \mathbb{N}$ such that for all $i \geq s$ we have $I_s = I_i$. And so $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_s = I$. Thus $I_s = I$. \hfill $\square$

**Definition 19.** Let $I \neq \langle 0 \rangle \subset R = k[x_1, x_2, \cdots, x_n]$ be an ideal and let $>$ be a term order. The leading term ideal of $I$ with respect to $>$ is

$$LT_>(I) = \langle LT_>(f) | f \in I \rangle.$$ 

Note that $LT_>(I)$ is a monomial ideal. By the above proposition there exist $f_1, f_2, \cdots, f_s \in I$ such that

$$LT_>(I) = \langle LT_>(f_1), LT_>(f_2), \cdots, LT_>(f_s) \rangle.$$ 

**Definition 20** (Grobner basis). Let $>$ be a monomial order on $R = k[x_1, x_2, \cdots, x_n]$ and let $I$ be a non-zero ideal of $R$. A finite subset $G = \{g_1, g_2, \cdots, g_s \}$ of $I$ is said to be a Grobner basis of $I$ with respect to the term order $>$ if

$$LT_>(I) = \langle LT_>(g_1), LT_>(g_2), \cdots, LT_>(g_s) \rangle.$$
**Proposition 2.** [4] Let \( I \subset R = k[x_1, x_2, \cdots, x_n] \) be an ideal and \( G = \{g_1, g_2, \cdots, g_s\} \) be a Grobner basis of \( I \) with respect to the term order \( > \). Then given \( f \in R \) there is a unique \( r \in R \) such that \( f = q_1g_1 + q_2g_2 + \cdots + q_sg_s + r \) and no term of \( r \) is in the ideal \( \langle LT>(g_1), LT>(g_2), \cdots, LT>(g_s) \rangle \).

**Proof.** By the division algorithm except \( r \) being unique everything else is true. So let us prove the uniqueness of \( r \). Suppose for a contradiction that \( r \) is not unique. Then \( f \) has two different representations, \( f = q_1g_1 + q_2g_2 + \cdots + q_sg_s + r_1 \) and \( f = q'_1g_1 + q'_2g_2 + \cdots + q'_sg_s + r_2 \).

Let \( q_1g_1 + q_2g_2 + \cdots + q_sg_s = f_1 \) and \( q'_1g_1 + q'_2g_2 + \cdots + q'_sg_s = f_2 \). Then we know that \( f_1, f_2 \in I \) and \( f_1 - f_2 \in I \implies r_2 - r_1 \in I \). However, by the definition of Grobner basis we can see that \( LT>(r_2 - r_1) \in J = \langle LT>(g_1), LT>(g_2), \cdots, LT>(g_s) \rangle \).

Since \( LT>(r_2 - r_1) \) is a monomial which is not zero and it is in the monomial ideal \( J \). This implies that at least one of the generating monomials must divide \( LT>(r_2 - r_1) \) by Lemma 6. If (WLOG) \( LM(r_1) > LM(r_2) \) then \( LT(r_1 - r_2) = LT(r_1) \) implies \( LT(r_1) \in J \), contradiction.

And if \( LM(r_1) = LM(r_2) \) then \( LM(r_1 - r_2) = LM(r_1)[LC(r_1) - LC(r_2)] \). This is in \( J \) implies that \( LT(r_1) \) is divisible by one of the generating monomials of \( J \), contradiction. Hence in both the cases it is a contradiction. So \( r_1 = r_2 \). \( \square \)

**Lemma 7.** [4] For any given ideal \( I \) and a term order \( > \) there is a Grobner basis \( G = \{g_1, g_2, \cdots, g_s\} \) such that \( \langle g_1, g_2, \cdots, g_s \rangle = I \).

**Proof.** \( LT>(I) \) is generated by a finite set of monomials \( x^{\alpha(1)}, x^{\alpha(2)}, \cdots, x^{\alpha(s)} \). By construction, there exists \( g_1, g_2, \cdots, g_s \in I \) such that \( LT>(g_i) = x^{\alpha(i)} \). It is clear that \( \langle g_1, g_2, \cdots, g_s \rangle \subset \)
I. We need to prove that $I$ is contained in $\langle g_1, g_2, \cdots, g_s \rangle$. Let $f \in I$. Then by the division algorithm we know that there exists a unique $r \in R$ such that $f = q_1g_1 + \cdots + q_sg_s + r$ where none of the terms of $r$ are divisible by $LT_>(g_i)$. Now we show that $r = 0$. Suppose that $r \neq 0$. Then $r = f - q_1g_1 - \cdots - q_sg_s$ and $LT_>(r) = LT_>(f - q_1g_1 - \cdots - q_sg_s)$. Since $f, g_1, \cdots, g_s \in I$ we know that $LT_>(r) = LT_>(f - q_1g_1 - \cdots - q_sg_s) \in \langle LT_>(g_1), LT_>(g_2), \cdots, LT_>(g_s) \rangle$ but this is a contradiction and hence $r = 0$ which implies that $f \in \langle g_1, g_2, \cdots, g_s \rangle$. 

Corollary 1. Let $G = \{g_1, \cdots, g_s\}$ be a Grobner basis of $I$ with respect to a term order $\succ$. Then

$$f \in I \iff f \text{ gives reminder zero upon division of elements in } G.$$ 

We have proved the existence of a Grobner basis of an ideal $I$ with respect to a term order $\succ$. The next question is: can we compute this Grobner basis? If so, is there an algorithm which gives us a procedure to compute it?

Definition 21 (S-polynomial). Let $f, g \in R = k[x_1, \cdots, x_n]$ be non-zero polynomials. Fix a term order $\succ$. Let $LM_>(f) = x^\alpha$ and let $LM_>(g) = x^\beta$. Define $x^\gamma = LCM(x^\alpha, x^\beta)$, where $\gamma_i = \max(\alpha_i, \beta_i)$. The S-polynomial of $f$ and $g$ is defined to be

$$S(f, g) = \frac{x^\gamma}{LT_>(f)}f - \frac{x^\gamma}{LT_>(g)}g.$$ 

Theorem 7. [4, Buchberger’s criterion] Let $G = \{g_1, \cdots, g_s\}$ be a generating set of an ideal $I$ in $R = k[x_1, \cdots, x_n]$. Then $G$ is a Grobner basis of $I$ with respect to the term order $\succ$ if
and only if for every pair \((i, j)\) where \(i \neq j\), \(S(g_i, g_j)\) has remainder zero when the division algorithm is applied using \(G\).

This theorem gives a way to check whether a generating set \(G\) of an ideal \(I\) is a Grobner basis for \(I\) or not. If such a generating set is not a Grobner basis then by this theorem there is a pair \((i, j)\) such that \(S(g_i, g_j)\) gives a non-zero remainder upon division by the elements of \(G\). Now if this is the case then we add \(r\) to the set \(G\) and check if the set \(G \cup \{r\}\) is a Grobner basis. Of course, \(G \cup \{r\}\) is a generating set but we need to check if it is a Grobner basis. If there is still some \(S\) polynomial that does give a non-zero remainder then we again iterate the process until all the \(S\)-polynomials give remainder zero. Then in that case, we found a Grobner basis of \(I\). This process is called Buchberger’s algorithm which we will be discussing next. We will also prove that this process terminates in a finite number of iterations.

**Buchberger’s algorithm**

Given the idea on how to check if a generating set \(G\) of an ideal \(I\) is a Grobner basis we are in the place to state the algorithm by which we compute the Grobner basis of an ideal \(I\).

**Corollary 2.** [4, Buchberger’s algorithm] Let \(I\) be an ideal of the polynomial ring \(R = k[x_1, \ldots, x_n]\). Let \(F = \{f_1, f_2, \ldots, f_n\}\) be a generating set of \(I\). Then the algorithm is as follows.

**Input:** \(F = \{f_1, f_2, \ldots, f_n\}\) and a term order >.
Output: A Grobner basis $G = \{g_1, \cdots, g_t\}$ for $I$ with respect to $>$. 

\[ G := F \]

\text{REPEAT}

\[ G' := G \]

\text{For each pair } \{f_i, f_j\}, f_i \neq f_j \text{ in } G' \text{ DO}

\[ r := \text{remainder of } S(f_i, f_j) \text{ by division by } G' \]

\text{IF } r \neq 0 \text{ THEN } G := G \cup \{r\}

\text{UNTIL } G = G'

It is clear that $G$ is a generating set of $I$ in each iteration of the algorithm. And also, when the algorithm terminates, the output $G$ is a Grobner basis of $I$ by Buchberger’s criterion. We claim that the algorithm will terminate. On each iteration performed when $r \neq 0$, we set $G = G' \cup \{r\}$ and thus $\langle \text{LT}(G') \rangle \subsetneq \langle \text{LT}(G) \rangle$. Suppose the algorithm does not terminate, then there is an infinite ascending chain of ideals that does not stabilize. This is a contradiction to the fact that $R$ is a Noetherian ring. Thus the algorithm must terminate.
Chapter 2

Minimal primes of adjacent tensors

In this chapter we will work with ideals of adjacent tensors which are a special class of binomial ideals. First, we will define a more general form of binomial ideals called lattice basis ideals. We will state the main results on their minimal primes based on Hosten and Shapiro’s work in [10].

Definition 22 (Lattice). A lattice $\mathcal{L}$ is a subgroup of the additive group $\mathbb{Z}^n$.

Example 11. $2\mathbb{Z}^n = \{(2z_1, 2z_2, \cdots, 2z_n) | (z_1, z_2, \cdots, z_n) \in \mathbb{Z}^n\} \subset \mathbb{Z}^n$ is a lattice.

Definition 23 (Saturated lattice). A lattice $\mathcal{L} \subset \mathbb{Z}^n$ is saturated if for $u \in \mathbb{Z}^n$ and $s \in \mathbb{Z}$, $su \in \mathcal{L}$ implies that $u \in \mathcal{L}$.

Throughout this thesis, we will assume that all lattices to be saturated and that there are no positive vectors in our lattice, i.e., $\mathcal{L} \cap \mathbb{N}^n = \{0\}$. 
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**Definition 24** (Toric ideal of \( L \)). Let \( L \subset \mathbb{Z}^n \) be a saturated lattice and \( R = k[x_1, x_2, \cdots, x_n] \).
Then the toric ideal of \( L \) is defined to be

\[
I_L = \langle x^\alpha - x^\beta : \alpha, \beta \in \mathbb{N}^n \text{ and } \alpha - \beta \in L \rangle.
\]

Remark: For any saturated lattice \( L \) the corresponding toric ideal \( I_L \) is prime [2].

**Definition 25.** Let \( u = (u_1, u_2, \cdots, u_n) \) be a vector in \( \mathbb{Z}^n \), then the positive components of \( u \) are \( \{u_{i_1}, u_{i_2}, \cdots, u_{i_t}\} \) such that \( u_{i_j} > 0 \) and the negative components of \( u \) are \( \{u'_{i_1}, u'_{i_2}, \cdots, u'_{i_t}\} \) such that \( u'_{i_j} < 0 \).

**Definition 26** (Lattice basis ideal). Let \( L \subset \mathbb{Z}^n \) be a saturated lattice and let \( B = \{u_1, u_2, \cdots, u_r\} \subset \mathbb{Z}^n \) be a basis for \( L \). We set \( f_{u_i} = x^{(u_i)_+} - x^{(u_i)_-} \in R[x_1, x_2, \cdots, x_n], \) where \( (u_i)_+ \) and \( (u_i)_- \) denote the positive and negative parts of \( u \), respectively. Then the ideal \( J_B = \langle f_{u_1}, f_{u_2}, \cdots, f_{u_r} \rangle \) is a lattice basis ideal.

We remark that, in general \( J_B \nsubseteq I_L \).

### 2.1 Adjacent tensor binomials

We will now define a class of binomial ideals called **tensor binomial ideals**. Then we will introduce **adjacent tensor binomial ideals** which will be the topic of study.

**Definition 27.**  \( \bullet \) For \( m \in \mathbb{N} \), let \([m] = \{1, 2, 3, \cdots, m\} \), and let \( \Gamma = [d_1] \times [d_2] \times \cdots [d_n] \) for \( d_i \in \mathbb{N} \). Consider the polynomial ring \( R_\Gamma = k[x_{i_1,i_2,\cdots,i_n} : (i_1, \cdots, i_n) \in \Gamma] \). Then the array \( T_\Gamma = [x_{i_1,\cdots,i_n} : (i_1, \cdots, i_n) \in \Gamma] \) is called a (generic) tensor.
• A 2-subtensor $U$ of $T_\Gamma$ is defined to be

$$U = [x_{i_1,\ldots,i_n} : (i_1, \ldots, i_n) \in I_1 \times I_2 \times \cdots \times I_n],$$

$I_i = \{p_i, q_i\} \subset [d_i]$ such that $p_i < q_i$.

• Given a 2-subtensor $U$ on $I = \{p_1, q_1\} \times \cdots \times \{p_n, q_n\}$, let $p = (p_1, p_2, \cdots, p_n)$ and define the set

$$A(U) = \{r \in I \mid \text{where an even number of components of } r \text{ is different from the components of } p\}.$$

Note that $p \in A(U)$. Let $B(U) = I \setminus A(U)$.

• The tensor binomial of a 2-subtensor $U$ on $I$ is

$$\prod_{r \in A(U)} x_r - \prod_{s \in B(U)} x_s.$$

**Definition 28** (Adjacent tensor binomial ideals). Let $\Gamma_{n,k} = [2] \times [2] \times \cdots \times [2] \times [k]$, and let $R_{n,k} = \mathbb{R}_{\Gamma_{n,k}} = k[[x_{i_1,i_2,\ldots,i_n} | (i_1, i_2, \cdots, i_n) \in \Gamma_{n,k}]]$.

Let $T_{i,i+1}$ be the 2-subtensor on $I_{i,i+1} = [2] \times [2] \times \cdots \times [2] \times \{i, i+1\}$. Such a 2-sub-tensor we will call an adjacent tensor.

Then the set of adjacent tensor binomials on $\Gamma_{n,k}$ is $\{\text{tensor binomial of } T_{i,i+1} | 1 \leq i \leq k-1\}$. We will denote the ideal generated by this set of tensor binomials by $I_{n,k}$ and call it an adjacent tensor binomial ideal.

Here, the lattice is the lattice generated by the exponent vectors of the tensor binomials $T_{i,i+1}$ and these vectors form a $\mathbb{Z}$-basis for that lattice. Hence the adjacent tensor binomial ideals are lattice basis ideals.
Example 12. Let $\Gamma_{2,5} = [2] \times [5]$ and let $R_{\Gamma_{2,5}} = k\{x_{ij}|(i, j) \in \Gamma_{2,5}\}$. Then the tensor $T_{2,5}$ is
\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\
x_{21} & x_{22} & x_{23} & x_{24} & x_{25}
\end{pmatrix}.
\]
The adjacent tensor binomial ideal $I_{2,5}$ is the ideal generated by the $2 \times 2$ adjacent minors of the above matrix:
\[
I_{2,5} = \langle x_{11}x_{22} - x_{21}x_{12}, x_{12}x_{23} - x_{22}x_{13}, x_{13}x_{24} - x_{23}x_{14}, x_{14}x_{25} - x_{24}x_{15} \rangle.
\]

\[
R_{3,4} = \mathbb{Q}[x_{111}, x_{112}, x_{113}, x_{114}, x_{121}, x_{122}, x_{123}, x_{124}, x_{211}, x_{212}, x_{213}, x_{214}, x_{221}, x_{222}, x_{223}, x_{224}].
\]
The tensor $T_{3,4}$ is shown in Figure 2.1,

![Figure 2.1: The tensor $T_{3,4}$](image)

The adjacent tensor binomial ideal is
\[
I_{3,4} = \langle \text{tensor binomials of } T_{1,2}, T_{2,3}, T_{3,4} \rangle.
\]
CHAPTER 2. MINIMAL PRIMES OF ADJACENT TENSORS

In expanded form $I_{3,4}$ is

$$\langle x_{11}x_{22}x_{12}x_{21} - x_{12}x_{21}x_{11}x_{22}, x_{112}x_{222}x_{213}x_{123} - x_{122}x_{212}x_{113}x_{223},$$

$$x_{113}x_{223}x_{124}x_{214} - x_{123}x_{213}x_{114}x_{224}\rangle.$$

Having defined the adjacent tensor binomial ideals, we will take a step back to the lattice basis ideals and state a few results about them. We saw earlier that if $J_B$ is a lattice basis ideal associated to a lattice $\mathcal{L}$ then in general $J_B \subset I_\mathcal{L}$. Moreover, we will define a basis matrix for lattice basis ideals. Surprisingly, the basis matrix gives us the minimal primes of the lattice basis ideals. Using this idea we will characterize the minimal primes of adjacent tensor binomial ideals.

**Definition 29 (Basis matrix).** Let $B = \{u_1, u_2, \cdots, u_r\} \subset \mathbb{Z}^n$ be a basis for a saturated lattice $\mathcal{L}$. We let

$$M(B) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{pmatrix}$$

and call it the basis matrix of $B$. Note that $M(B)$ is an $r \times n$ matrix.

**Example 14.** Let us consider Example 12, where

$$I_{2,5} = \langle x_{11}x_{22} - x_{21}x_{12}, x_{12}x_{23} - x_{22}x_{13}, x_{13}x_{24} - x_{14}x_{23}, x_{14}x_{25} - x_{24}x_{15}\rangle.$$

The basis matrix of this lattice generating $I_{2,5}$ is
Our goal is to characterize the minimal primes of a lattice basis ideal $J_B$ in terms of the basis matrix $M(B)$. And to do that we define mixed, unmixed and dominating matrices. Furthermore, we introduce irreducible sub-matrices of $M(B)$ and state the theorem on how these correspond to the minimal primes of $J_B$. At last, we use this theorem to characterize the minimal primes of the adjacent tensor binomial ideals.

**Definition 30** (Mixed, unmixed and dominating matrices). Let $M$ be an $r \times n$ matrix.

- $M$ is called a **mixed** matrix if every row of $M$ has both a positive and a negative entry, otherwise it is called **unmixed**.

- The matrix $M$ is called **dominating** if it is mixed, but no $t \times t$ submatrix of $M$ is mixed for $1 \leq t \leq \min(r, n)$.

**Theorem 8.** [10] Let $B$ be a basis for a saturated lattice $\mathcal{L} \subset \mathbb{Z}^n$, and let $M(B)$ be the basis matrix of $B$. Then $M(B)$ is dominating if and only if $J_B = I_L$.

**Proposition 3.** [10] Let $B$ be a basis for a saturated lattice $\mathcal{L} \subset \mathbb{Z}^n$. Then $I_L$ is a minimal prime of $J_B$ and every minimal prime $P$ of $J_B$ other than $I_L$ has the form $P =$
\langle x_{i_1}, x_{i_2}, \cdots, x_{i_t}, I_{L'} \rangle$, where $t \geq 1$ and $L'$ is the lattice generated by the basis vectors $u_{j_1}, \cdots, u_{j_k}$ such that $\text{supp}(u_{j_p}) \cap \{i_1, \cdots, i_t\} = \emptyset$ for $p = 1, \cdots, k$. Thus $P$ is determined by the set of variables contained in it.

Definition 31. A matrix $N$ is irreducible if:

- $N$ is a mixed $s \times t$ matrix where $t \leq s$, and
- one cannot bring $N$ into the following form after permuting its rows and columns:

$$N = \begin{pmatrix} N_0 & B_0 \\ 0 & D_0 \end{pmatrix}$$

where $N_0$ is mixed and $D_0$ is of size $m \times p$ where $m < p$.

Definition 32. Let $N$ be a irreducible sub-matrix of the basis matrix $M(B)$ and suppose after row and column permutations $M(B) = \begin{pmatrix} N & B \\ 0 & D \end{pmatrix}$. The prime ideal $P_N$ is defined to be $\langle x_{i_1}, x_{i_2}, \cdots, x_{i_t}, I_{L'} \rangle$, where $x_{i_1}, x_{i_2}, \cdots, x_{i_t}$ are the variables which correspond to the columns of $N$ and $I_{L'}$ is the toric ideal corresponding to the lattice generated by the rows of $D$.

Theorem 9. [10] Let $P$ be a prime ideal containing at least one variable induced by the rearrangement of the basis matrix $M(B)$. Then $P$ is a minimal prime of $J_B$ if and only if $P = P_N$ for some irreducible $N$.

Thus for a lattice basis ideal $J_B$ if we need to characterize its minimal primes, it is sufficient to characterize the irreducible sub-matrices of the basis matrix $M(B)$. 

Example 15. Let us consider the adjacent tensor binomial ideal \( I_{2,4} \). It is generated by the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24}
\end{pmatrix}.
\]

So

\[ I_{2,4} = (x_{11}x_{22} - x_{21}x_{12}, x_{12}x_{23} - x_{22}x_{13}, x_{13}x_{24} - x_{14}x_{23}). \]

The basis matrix of the lattice \( B \) generating \( I_{2,4} \) is \( M(B) \)

\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} \\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

The irreducible sub-matrices \( N \) of \( M(B) \) and the corresponding minimal primes \( P_N \) of \( I_{2,4} \) are described below.

So \( (x_{13}, x_{23}, x_{11}x_{22} - x_{21}x_{12}) \) is a minimal prime and the corresponding rearrangement of the basis matrix is

\[
\begin{pmatrix}
x_{23} & x_{13} & x_{11} & x_{12} & x_{14} & x_{21} & x_{22} & x_{24} \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & -1 & 1 & 0
\end{pmatrix}
\]
Similarly, \((x_{12}, x_{22}, x_{13}x_{24} - x_{13}x_{14})\) is a minimal prime and the corresponding rearrangement of the basis matrix is

\[
\begin{pmatrix}
  x_{22} & x_{12} & x_{11} & x_{13} & x_{14} & x_{21} & x_{23} & x_{24} \\
  1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\
  -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 & -1 & 0 & -1 & 1
\end{pmatrix}
\]

**Remark:** For a tensor, \(T_{n,k}\) the number of generators for the adjacent tensor binomial ideal \(I_{n,k}\) is \(k - 1\) and the number of variables in the ring \(R_{n,k}\) is \(2^{n-1}k\). If \(\mathcal{B}\) is a basis of the lattice generating \(I_{n,k}\), then \(M(\mathcal{B})\) is a \((k - 1) \times (2^{n-1}k)\) matrix.

**Theorem 10.** The adjacent tensor binomial ideal \(I_{n,k}\) is a radical ideal. Therefore, the primary decomposition of \(I_{n,k}\) gives us the minimal primes.

The proof of this will be discussed in the next chapter where we prove the same result for a more general class of binomial tensor ideals.

**Definition 33.** Let \(m\) be an even positive integer. Then \(V_m\) is defined to be the matrix of dimension \(1 \times m\) of the form

\[
\begin{pmatrix}
  1 & -1 & 1 & -1 & \cdots & \cdots & 1 & -1
\end{pmatrix}.
\]
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Theorem 11. 1. The basis matrix \( M(\mathcal{B}) \) associated to the adjacent tensor binomials generating \( I_{n,k} \) can be brought to the form

\[
M(\mathcal{B}) = \begin{pmatrix}
V_{2n-1} & V_{2n-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & -V_{2n-1} & -V_{2n-1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & V_{2n-1} & V_{2n-1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & (-1)^{k-2}V_{2n-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & (-1)^{k-1}V_{2n-1} & (-1)^{k-1}V_{2n-1}
\end{pmatrix}
\]

by permuting its rows and columns.

2. The irreducible sub-matrices of this basis matrix are of the form \( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \), \( \begin{pmatrix}
A & 0 & 0 \\
0 & A & 0 \\
0 & 0 & A
\end{pmatrix} \) etc., where \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \).

We will state and prove some lemmas that will lead us to the proof of this theorem.

Lemma 8. Let \( M(\mathcal{B}) \) and \( A \) be as in Theorem 11. Then for any \( r \in \mathbb{N} \) the matrix

\[
A_{2r} = \begin{pmatrix}
A & 0 & 0 & \cdots & 0 \\
0 & A & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 \\
0 & \cdots & \cdots & \cdots & A
\end{pmatrix}
\]
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is an irreducible sub-matrix of $M(B)$.

**Proof.** The matrix $A$ is irreducible by Definition 31. Let $r > 1$ and we will prove that $A_{2r}$ is irreducible. It is clear that the condition 1 in Definition 31 is true. Let us suppose we can rearrange the rows and columns such that, $A_{2(t+1)} = \begin{pmatrix} N_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$ where $N_0$ is mixed and $D_0$ is of size $m \times p$ where $m < p$ and all the entries of $C_0$ are zero. Since each row and column of $A_{2r}$ had exactly two non-zero entries $(+1, -1)$, so is the new rearranged matrix. Since $N_0$ is mixed, all the rows of $N_0$ take the two non-zero elements. Hence all the elements of $B_0$ must be zero. And since all the entries of $C_0$ are zero, in each row of $D_0$ there must be exactly two non-zero elements. Because $D_0$ has more columns than rows, and $A_{2r}$ is a square matrix, we see that $N_0$ will have more rows than columns. Then by symmetry either $B_0$ has a non-zero entry or $C_0$ has a non-zero entry in it. But this is a contradiction.

**Lemma 9.** Let $M(B)$ be the basis matrix stated in Theorem 11. If $N$ is an irreducible sub-matrix of $M(B)$ then each column of $N$ has exactly two non-zero entries.

**Proof.** Since each column of $M(B)$ has at most two non-zero entries, it is enough to prove that if $N$ is an irreducible matrix then $N$ does not have a column with exactly one non-zero entry. Suppose there is a column $i$ of $N$ which has exactly one non-zero element, say 1. Let us denote the row by $j$ so $N_{ji} = 1$. And since $N$ is mixed there is another column $i'$ such that $N_{ji'} = -1$. Suppose that there is another non-zero entry in the $j$th row other than
these two. Then $N$ has one of the following sub-matrices in it:

\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & -1 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & -1 & 1 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

It is clear that $N$ cannot be any of these matrices, because if it is, then in the first three cases $N$ has more columns than rows and in the last case $N$ is not mixed. So $N$ should have other columns. Observing the first three matrices it is clear that no matter what the other sub-matrix is, we have already more columns than rows. So $N$ will have more columns than rows, a contradiction. In the last case, if we need to make the last row mixed then two things can happen:

\[
\begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

In both of these cases we will have more columns than rows. In particular, to make the last row of the second matrix mixed we need to add one more column, which will make the number of columns greater than the number of rows. So $N$ is not irreducible, a contradiction. We conclude that if there exists a column with exactly one non-zero entry, say $N_{ji}$, then the $j$th row has exactly 2 non-zero entries.

- Let us suppose that the $i'$th column also has exactly one non-zero entry $N_{ji'} = -1$. 

Then we can rearrange $N$ such that $N = \begin{pmatrix} N_0 & B_0 \\ 0 & -1 & 1 \end{pmatrix}$. Here $N_0$ is not empty because if it is then $N = (1 \ -1)$, and $N$ is not irreducible in the first place. Thus $N_0$ is non-empty and should be mixed. This contradicts the irreducibility of $N$.

- Let us suppose that there is another row $j' \neq j$ such that $N_{j'i'}$ is non-zero. By the property of the basis matrix $N_{j'i'} = 1$. Now there should be another column that makes the $j'$th row mixed. Take such a column $i''$. We know that $N_{ji''} = -1$ is non-zero. If $N_{ji} = 1$ then we arrive at a contradiction to the irreducibility of $N$ because then using the same argument as above we can show that $N$ will have more columns than rows.

This proves that no column of $N$ has exactly one non-zero entry. Thus it should have exactly two non-zero entries. \hfill \Box

The next lemma is identical to Lemma 9 but with the columns replaced by rows.

**Lemma 10.** Let $M(B)$ be the basis matrix stated in Theorem 11. If $N$ is a sub-matrix of $M(B)$ then each row of $N$ will have exactly 2 non-zero entries.

**Proof.** Let us suppose $N$ has a row with more than 2 non-zero entries. Then we have four cases. The sub-matrix of these three columns will take one of these possibilities,
The first matrix cannot be a possible sub-matrix of $M(B)$. Because each adjacent subtensors can overlap with at most two other adjacent tensors. Here the tensors corresponding to the first row overlaps with three other tensors. This is not possible.

- In all the other three cases it is clear that $N$ cannot be these matrices. And so there are other columns to $N$. By using the same arguments as before $N$ will have more columns than rows. So all three cases are not possible.

Hence we get a contradiction to the fact that $N$ is irreducible. We conclude that if $N$ is irreducible each row should have exactly two non-zero entries. 

\[ \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} ; \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \]

Proof. Theorem 11. It is clear that statement 1 of Theorem 11 is true. By Lemma 8 all matrices mentioned in statement 2 are irreducible. And by Lemma 9 and Lemma 10 if $N$ is irreducible then each column and each row of $N$ should have exactly two non-zero entries. And by statement 1 from Theorem 11, it is clear that $N$ is of the form mentioned in statement 2.

In the previous theorem, we saw that to characterize the minimal primes of the adjacent tensor binomial ideals we just need to look for the irreducible matrices of the basis matrix.
Since we have characterized the irreducible matrices by the above theorem, we can count the number of minimal primes of a given $T_{n,k}$. The next theorem gives us a counting formula for the number of minimal primes.

**Theorem 12.** The number of minimal primes of the adjacent tensor binomial ideal $I_{n,k}$ is

$$1 + \sum_{s=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor} J_s (2^{2n-4})^s$$

where $J_s = \binom{k-(s+1)}{s}$.

**Proof.** The basis matrix has dimensions $(k - 1) \times (2^n - 1) k$, and each matrix $A$ is $2 \times 2$. Let us first consider the set $\Omega = \{(1, 2), (2, 3), \ldots, (k - 2, k - 1)\}$ which indexes the set of rows of the basis matrix after rearranging them to the form mentioned in Theorem 11. Each $A$ could be chosen from any of these sets. More precisely we need to choose the adjacent rows in order to construct $A$. However, there are some cases where we need to choose more than one $A$ matrix, say for example, for the irreducible matrix $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ we are choosing two $A$ matrices. In this case, suppose the first $A$ matrix is from row $(3, 4)$ then the second $A$ matrix cannot be $(2, 3)$ or $(4, 5)$. If we are choosing more than one element from $\Omega$ then the neighbors should be avoided.

To solve this problem let us state a similar problem from combinatorics.

Given $k - s$ stars, in how many ways can we place $s$ bars in between these stars such that at most 1 bar can be placed between two stars and no bars can be placed at the end and at the beginning? To illustrate this let us choose $k = 7$ and $s = 3$ then we are asked the question, given $7 - 3 = 4$ stars in how many ways can we place 3 bars in between the stars? There is
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exactly one possibility

\[ \times \big| \times \big| \times \big| \times \]

In general given \( k - s \), the answer to the above question is \( J_s = \binom{k-s-1}{s} \). Given \( k - s \) stars there are exactly \( k - s - 1 \) gaps to place the bars. And the number of ways to place \( s \) bars is precisely \( J_s \).

Let us consider the set \( \Omega \) again. We will make a slight change to \( \Omega \), instead of \( \Omega = \{(1, 2), (2, 3) \cdots, (k - 2, k - 1)\} \) we include two more elements. Let

\[ \Omega = \{(0, 1), (1, 2), (2, 3), \cdots, (k - 2, k - 1), (k - 1, k)\} \]

Then the rule is that we cannot pick \((0, 1)\) and \((k - 1, k)\). The number of ways we pick \( s \) elements in \( \Omega \) such that the neighbors must be avoided is \( J_s \).

Let us go back to the example we illustrated before. Let \( k = 7 \) and \( s = 3 \), given \( 7 - 3 = 4 \) stars in how many ways can we place \( 3 \) bars in between the stars? There is exactly one possibility which is,

\[ \times \big| \times \big| \times \big| \times \big| \times \]

\((0, 1) \quad (1, 2) \quad (2, 3) \quad (3, 4) \quad (4, 5) \quad (5, 6) \quad (6, 7)\)

So in this setting, we have chosen the elements which correspond to the bars. We chose \(\{(1, 2), (3, 4), (5, 6)\}\). Thus we have shown what \( J_s \) is contributing in the formula. Here we need to take into account the other possibilities of \( s \) as well. As we can see when \( s = 2 \) we have \( 7 - 2 = 5 \) stars and we need to place \( 2 \) bars in between the stars. There are \( \binom{4}{2} = 6 \) possibilities. And if we set \( s = 4 \), then \( k - s = 3 \) and there are \( 4 \) bars to be placed in 2
spaces, which is not possible according to our rules. So we take all the \( s \) which is in the range \( 1 \leq s < \left\lfloor \frac{k}{2} \right\rfloor \) because \( k - \left\lfloor \frac{k}{2} \right\rfloor - 1 < \left\lfloor \frac{k}{2} \right\rfloor \) and we cannot place \( \left\lfloor \frac{k}{2} \right\rfloor \) things in \( k - \left\lfloor \frac{k}{2} \right\rfloor - 1 \) places with our conditions.

After we have chosen the rows there are still many possibilities for choosing the columns for the \( A \) matrix. Since for each chosen element from \( \Omega \), there is a corresponding matrix of the form
\[
\begin{pmatrix}
I_{2^n-1} \\
-I_{2^n-1} \\
-I_{2^n-1} \\
I_{2^n-1}
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
-I_{2^n-1} \\
I_{2^n-1}
\end{pmatrix}
\]. Let us suppose that it takes the first form. Then we need to choose exactly two columns from it such that one column is \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and the other column is \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \). Since there are \( 2^{n-1} \) elements in total and out of which half are 1 and the other half are \(-1\), so we have \( 2^{n-2} \) elements equal to 1 and \( 2^{n-2} \) elements equal to \(-1\). Now we need to choose one from each so we have \( 2^{n-2} \) possibilities, which is \( 2^{2n-4} \). Depending on the \( s \) value the number of possibilities increases. If \( s = 1 \) then we have \( 2^{2n-4} \) and if \( s = 2 \) we have \( 2^{2n-4} \) for one row and \( 2^{2n-4} \) for the other row, and similarly for any arbitrary \( s \) we have \( (2^{2n-4})^s \) possibilities. For each \( s \) there are \( J_s \) possible choices of rows and so for each \( s \) we have \( (2^{2n-4})^s J_s \) possibilities and in total we sum over all the possible \( s \) values. Hence we get the desired result
\[
\sum_{s=1}^{\left\lfloor \frac{k-2}{2} \right\rfloor} J_s (2^{2n-4})^s \]
where \( J_s = \binom{k-(s+1)}{s} \). If we include the toric part in it we get one additional minimal prime.
Chapter 3

Binomial (tensor) edge ideals

3.1 Binomial tensor edge ideals

In the previous section, we studied adjacent tensor binomial ideals. For a tensor $T_{ \Gamma_n,k}$, the adjacent tensor binomials were constructed using the binomials from the 2-sub-tensors of the form $T_{i,i+1}$. A graph $G$ with edges only between consecutive terms is called a $k$-path graph, i.e, the edge set of $G$ is $\{(1,2), (2,3), \ldots, (k-1,k)\}$. If we construct binomials from the 2-subtensors of the form $T_{i,j}$ such that $(i,j)$ is in the edge set of a $k$-path graph, then the ideal generated by those binomials is precisely the adjacent tensor binomial ideal. In general, for any simple graph $G$ we can construct a binomial ideal of 2-tensors in $T_{ \Gamma_n,k}$. This will be the topic of this chapter. For any simple graph $G$ on $k$ vertices we will define what is called a binomial tensor edge ideal of $G$, and we will characterize its minimal primes. If we consider the tensor $T_{ \Gamma_2,k}$ and define an ideal in a similar way for any simple graph $G$
then this ideal is called a binomial edge ideal and was studied in [9]. Recall that $T_{\Gamma_{2,k}}$ is a $2 \times k$ matrix of indeterminants. Many things are known about them including their minimal primes and that they are Cohen-Macaulay etc. We will extend the result on minimal primes of the binomial edge ideals to the binomial tensor edge ideals.

**Definition 34.** Let $\Gamma_{n,k} = [2] \times [2] \times \cdots \times [k]$, and let $T_{\Gamma_{n,k}} = \{x_{i_1,\ldots,i_n} : (i_1, \ldots, i_n) \in \Gamma_{n,k}\}$ be a generic tensor on $\Gamma_{n,k}$. Further let $G$ be a simple graph on $k$ vertices, and let $R_{n,k} = K[\{x_{i_1,i_2,\ldots,i_n} | (i_1, i_2, \ldots, i_n) \in [2] \times [2] \times \cdots \times [k]\}]$ be a polynomial ring over a field $K$. Let $T_{i,j}$ be the 2-sub-tensor of $T_{\Gamma_{n,k}}$ on

$$U(i,j) = \underbrace{\{1,2\} \times \{1,2\} \times \cdots \{1,2\} \times \{i,j\}}_{n \text{ times}},$$

let $f_{ij}$ be the tensor binomial of $T_{ij}$. Then the binomial tensor edge ideal of $G$ is

$$I_{n,k}(G) = \langle \{f_{ij} | \{i,j\} \text{ is an edge in } G \text{ and } i < j \} \rangle.$$ 

More precisely,

$$I_{n,k}(G) = \langle \{ \prod_{r \in A(U_{i,j})} x_r - \prod_{s \in B(U_{i,j})} x_s | \{i,j\} \text{ is an edge in } G \} \rangle,$$

where $A(U_{i,j}) = A_{i,j}$ and $B(U_{i,j}) = B_{i,j}$.

Recall, given a 2-subtensor $U$ on $I = \{p_1,q_1\} \times \cdots \times \{p_n,q_n\}$, let $p = (p_1,p_2,\ldots,p_n)$ then

$$A(U) = \{r \in I \text{ where an even number of components of } r \text{ is different from the components of } p\}.$$ 

And $B(U) = I \setminus A(U)$. 

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Example 16. Let $T_{3,4}$ be the tensor on $[2] \times [2] \times [4]$,

$$R_{3,4} = \mathbb{Q}[x_{111}, x_{112}, x_{113}, x_{114}, x_{121}, x_{122}, x_{123}, x_{124}, x_{211}, x_{212}, x_{213}, x_{214}, x_{221}, x_{222}, x_{223}, x_{224}],$$

and $G$ be a cycle on 4 vertices with edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Then

$$I_{3,4}(G) = \langle x_{111}x_{221}x_{122}x_{212} - x_{121}x_{211}x_{112}x_{222}, x_{112}x_{222}x_{213}x_{123} - x_{122}x_{212}x_{113}x_{223},$$

$$x_{113}x_{223}x_{124}x_{214}, x_{123}x_{213}x_{114}x_{224}, x_{111}x_{221}x_{124}x_{214} - x_{121}x_{211}x_{114}x_{224}\rangle$$

Definition 35. Let $\Gamma_{n,k} = [2] \times [2] \times [2] \times \cdots \times [k]$, and let $T_{\Gamma_{n,k}}$ be a tensor on $\Gamma_{n,k}$. For each $i \in [k]$ we define the $i$th slice of $\Gamma_{n,k}$ as,

$$\text{Slice}(i) = \underbrace{[2] \times [2] \times [2] \times \cdots \times [2]}_{(n-\text{times})} \times \{i\}.$$

For each Slice$(i)$ we define a tensor $T_i = \{x_r | r \in \text{Slice}(i)\}$.

Further, let $A_i = A_{i,j} \cap \text{Slice}(i)$ and $B_i = B_{i,j} \cap \text{Slice}(i)$ for any $j$.

Further for each $i \in [k]$, we define a set of two-element sets,

$$D(i) = \{\{x_r, x_s\} | x_r \in A_i \text{ and } x_s \in B_i\}.$$

Let $S = \{s_1, s_2, \cdots, s_t\} \subset [k]$, and for each $S \subset [k]$ we define the set

$$L(S) = \bigcup_{j=1}^{t} d_{s_j} | (d_{s_1}, \cdots, d_{s_t}) \in D(s_1) \times \cdots \times D(s_t)\}.$$

Further, we define $x_{A_i} = \{x_r | r \in A_i\}$ and similarly $x_{B_i} = \{x_s | s \in B_i\}$.

At last let $x^{A_i} = \prod_{r \in A_i} x_r$ and similarly $x^{B_i} = \prod_{s \in B_i} x_s$. 
Example 17. Let $T_{\Gamma_{3,4}}$ be the tensor on $\Gamma_{3,4} = 2 \times 2 \times 4$. Let $i = 3$, and $S = \{2, 3\}$, then

- $T_3 = \{x_{113}, x_{123}, x_{213}, x_{223}\}$.
- $A_3 = \{113, 223\}$ and $B_3 = \{123, 213\}$.
- $x_{A_3} = \{x_{113}, x_{223}\}$ and $x_{B_3} = \{x_{123}, x_{213}\}$.
- $x^{A_3} = x_{113}x_{223}$ and $x^{B_3} = x_{123}x_{213}$.
- $D(3) = \{\{x_{113}, x_{123}\}, \{x_{113}, x_{213}\}, \{x_{223}, x_{123}\}, \{x_{223}, x_{213}\}\}$.
- $D(2) = \{\{x_{112}, x_{122}\}, \{x_{112}, x_{212}\}, \{x_{222}, x_{122}\}, \{x_{222}, x_{212}\}\}$.
- $L(\{2, 3\})$ is

\[
\begin{align*}
\{\{x_{113}, x_{123}, x_{112}, x_{122}\}, & \{x_{113}, x_{123}, x_{112}, x_{122}\}, \{x_{113}, x_{123}, x_{222}, x_{122}\}, \\
\{x_{113}, x_{123}, x_{222}, x_{121}\}, & \{x_{113}, x_{123}, x_{112}, x_{122}\}, \{x_{113}, x_{213}, x_{112}, x_{122}\}, \\
\{x_{113}, x_{213}, x_{112}, x_{122}\}, & \{x_{113}, x_{213}, x_{222}, x_{122}\}, \{x_{223}, x_{123}, x_{112}, x_{122}\}, \\
\{x_{223}, x_{123}, x_{112}, x_{122}\}, & \{x_{223}, x_{123}, x_{222}, x_{122}\}, \{x_{223}, x_{213}, x_{112}, x_{122}\}, \\
\{x_{223}, x_{213}, x_{222}, x_{122}\}, & \{x_{223}, x_{213}, x_{222}, x_{122}\}, \{x_{223}, x_{213}, x_{222}, x_{122}\}, \\
\{x_{223}, x_{213}, x_{222}, x_{122}\}\}.
\end{align*}
\]

Observation: Let $G$ be a simple graph on $[k]$. If $(i, j)$ is an edge of $G$ such that $i < j$, then the binomial corresponding to $T_{i,j}$ is

\[
\prod_{r \in A_i} x_r \prod_{s \in B_j} x_s - \prod_{s \in B_i} x_s \prod_{r \in A_j} x_r = x^{A_i} x^{B_j} - x^{A_j} x^{B_i}.
\]
Definition 36 (Admissible path). Let $G$ be a simple graph on $[k]$, and let $i, j$ be vertices of $G$ with $i < j$. Then a path $i = i_0, i_1, \ldots, i_d = j$ from $i$ to $j$ is admissible if

- $i_p \neq i_q$ for $p \neq q$;

- for each $p = 1, \ldots, d - 1$ one has either $i_p < i$ or $i_p > j$;

- for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{d-1}\}$ the sequence $i, j_1, \ldots, j_s, j$ is not a path.

Given an admissible path $\pi : i = i_0, i_1, \ldots, i_d = j$, from $i$ to $j$, we associate the monomial

$$u_\pi = \prod_{i_p > j} x^{A_{i_p}} \prod_{i_q < i} x^{B_{i_q}}.$$

Definition 37. Let $G$ be a simple graph on $[k]$, let $T_{\Gamma_{n,k}}$ be a tensor on $\Gamma_{n,k} = (2 \times 2 \times 2 \times \cdots \times [k])^{(n-1) \text{ times}}$, and let $R_{n,k}$ be the polynomial ring over a field with variables from $T_{\Gamma_{n,k}}$. We can think of any element of $\Gamma_{n,k}$ as $(b_1, b_2, \ldots, b_{n-1}, i)$, where $b_m$ take values either 1, 2 and $i \in [k]$. And thus $b = (b_1, \ldots, b_{n-1}) \in (2 \times 2 \times 2 \times \cdots \times 2)^{(n-1) \text{ times}}$. Let $> \text{ be the lexicographic term order on } R_{n,k}$ induced by the indices of the variables. Note that for any $b, b' \in (2 \times 2 \times 2 \times \cdots \times 2)^{(n-1) \text{ times}}$ and $i, j \in [k]$ the term order $>$ is induced by

- $x_{b,i} > x_{b',j}$ if the left most non-zero component of $(b_1 - b_1', \ldots, b_{n-1} - b_{n-1}')$ is $-1$, where $b = (b_1, b_2, \ldots, b_{n-1})$ and $b' = (b_1', b_2', \ldots, b_{n-1}')$.

- In the case $b = b'$ then $x_{b,i} > x_{b,j}$ if $i < j$.  

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Theorem 13. Let $G$ be a simple graph on $[k]$, and let $>$ be the monomial order introduced above. Then the set of binomials

$$
G = \bigcup_{i<j} \{u_{\pi} f_{ij} \mid \pi \text{ is an admissible path from } i \text{ to } j\},
$$

is a Grobner basis for the tensor binomial ideal $I_{n,k}(G)$.

We first state and prove a short lemma which will lead us to the proof of this theorem.

Lemma 11. Let $G$ be a simple graph on $[k]$ and let $>$ be the monomial order introduced in Definition 37. Then the set of binomials

$$
G = \bigcup_{i<j} \{u_{\pi} f_{ij} \mid \pi \text{ is an admissible path from } i \text{ to } j\}
$$

is a subset of $I_{n,k}(G)$

Proof. Let us take any admissible path $\pi : i = i_0, i_1, \cdots, i_d = j$. Let us induct on $d$ to show that $u_{\pi} f_{ij} \in I_{n,k}(G)$. When $d = 1$ this is true because then $u_{\pi} = 1$ and we have $f_{ij} \in I(G)$, since $(i, j)$ is an edge of $G$. We proceed with the induction on $d$. If $d > 1$ we should have either $U = \{i_p | i_p < i\}$ or $V = \{i_q | i_q < j\}$ non-empty. If $U \neq \emptyset$ then we set $i_{p_0} = \max$ of $U$. Or if $V \neq \emptyset$ then we set $i_{q_0} = \max$ of $V$. We suppose $U \neq \emptyset$ and take two paths $\pi_1 = i_{p_0}, i_{p_0-1}, \cdots, i_0 = i$ and $\pi_2 = i_{p_0}, i_{p_0+1}, \cdots, i_d = j$. We claim that these paths are admissible. Clearly condition 1 and condition 3 from Definition 36 must hold. Condition 2 holds because all the elements $i_1, \cdots, i_{p_0-1}, i_{p_0+1} \cdots, i_{d-1}$ are either greater than $j$ or less than $i_{p_0}$ by construction.
By the induction hypothesis, we know that $u_{\pi_1}f_{ip_0}$ and $u_{\pi_2}f_{ip_0}$ are in $I(G)$. We claim that the $S$-polynomial $S(u_{\pi_1}f_{ip_0}, u_{\pi_2}f_{ip_0}) = -u_{\pi_1}f_{ij}$. Then $u_{\pi_1}f_{ij} \in I(G)$. We first compute the $S$-polynomial for this. Let $x^\alpha = LM_>(u_{\pi_1}f_{ip_0}) = u_{\pi_1}x^{A_{ip_0}x^{B_i}}$ and let $x^\beta = LM_>(u_{\pi_2}f_{ip_0}) = u_{\pi_2}x^{A_{ip_0}x^{B_j}}$. And let $x^\gamma = LCM(x^\alpha, x^\beta)$. As $u_{\pi_1}$ and $u_{\pi_2}$ has no common terms. We have,

$$x^\gamma = LCM(u_{\pi_1}x^{A_{ip_0}x^{B_i}}, u_{\pi_2}x^{A_{ip_0}x^{B_j}}) = u_{\pi_1}u_{\pi_2}x^{A_{ip_0}x^{B_i}x^{B_j}}.$$

Thus

$$S(u_{\pi_1}f_{ip_0}, u_{\pi_2}f_{ip_0}) = u_{\pi_2}u_{\pi_1}(x^{B_i}f_{ip_0} - x^{B_j}f_{ip_0}) = x^{B_{ip_0}}u_{\pi_1}u_{\pi_2}(-x^{B_i}x^{A_i} + x^{A_j}x^{B_i}) = -u_{\pi_1}f_{ij}.$$

\[ \square \]

With the Lemma 11, we can prove Theorem 13.

**Proof.** Theorem 13. By Lemma 11, we showed that $\mathcal{G} \subset I(G)$. So it is left to show that if $\pi$ and $\sigma$ are admissible paths from $i$ to $j$ and $m$ to $l$, respectively, then the $S$-polynomial $S(u_{\pi}f_{ij}, u_{\sigma}f_{ml})$ reduces to zero with respect to $\mathcal{G}$. We have a few cases to consider.

- If $i = m$ and $j = l$ then $S(u_{\pi}f_{ij}, u_{\sigma}f_{ml}) = 0$.

- If $\{i, j\} \cap \{m, l\} = \emptyset$ or $i = l$, or $j = m$, then $LM_>(f_{ij})$ and $LM_>(f_{kl})$ form a regular sequence. A regular sequence is a set of monomials $\{f_1, \ldots, f_r\}$ satisfying $gcd(f_i, f_j) = 1$ for all $i \neq j$. In our case, since there is no common term this sequence exists. In general, if $f, g \in \mathcal{G}$ form a regular sequence then for any monomials $u, v$, $S(uf, vg)$ reduces to zero under $\mathcal{G}$. So $S(u_{\pi}f_{ij}, u_{\sigma}f_{ml})$ reduces to zero under $\mathcal{G}$. 
• Now either \( i = m \) and \( j \neq l \) or \( i \neq m \) and \( j = l \). We prove the statement in the first case and the proof of the second case is similar. Consider two admissible paths, 
\( \pi : i = i_0, \ldots, i_p = j \) and \( \sigma : i = i'_0, i'_1, \ldots, i'_q = l \) where \( j < l \). Then there exist unique indices \( d, d' \) such that 
\[ i_d = i'_d \quad \text{and} \quad \{ i_{d+1}, \ldots, i_p \} \cap \{ i'_{d+1}, \ldots, i'_q \} = \emptyset. \]

Having this we have a path \( \tau \) from \( j \) to \( l \) given by 
\[ \tau : j = i_p, i_{p-1}, \ldots, i_{d}, i_{d} = i'_d, i'_{d+1}, \ldots, i'_{q-1}, i'_{q} = l. \]

To simplify the notation we write this as \( \tau : j = j_0, j_1, \ldots, j_{t-1}, j_t = l. \) We define 
\( j_{t(1)} = \min\{ j_c : j_c > j, c = 1, 2, \ldots, t \} \) and \( j_{t(2)} = \min\{ j_c : j_c > j, c = t(1) + 1, \ldots, t \} \) and so on. If we continue in this procedure we will get an ascending chain of integers,
\[ 0 = t(0) < t(1) < t(2) < \cdots < t(m) = t. \]
It then follows that \( j = j_{t(0)} < j_{t(1)}, \ldots, j_{t(m)} = l \) and for each \( 1 \leq c \leq t \), the path \( \tau_c : j_{t(c-1)}, j_{t(c-1)+1}, \ldots, j_{t(c)} \) is an admissible path.

We need to show that 
\[ S(u_\pi f_{ij}, u_\sigma f_{il}) = \sum_{c=1}^{q} v_{\tau_c} u_{\tau_c} f_{j_{t(c-1)}j_{t(c)}} \]
is a standard expression of \( S(u_\pi f_{ij}, u_\sigma f_{il}) \) whose remainder is zero. Here each \( v_{\tau_c} \) is the monomial defined as follows. First, let \( w = -x^{B_1 \text{lcm}(u_\tau, u_\sigma)} \).

1. If \( c = 1 \), then 
\[ v_{\tau_1} = \frac{x^{A_1 j_1}}{u_{\tau_1} x^{A_{t(1)}}}; \]
2. if \( c \in (1, q) \), then
\[
v_{x_c} = \frac{x^{A_j}x^{A_i}w}{u_{x_c}x^{A_j(e-1)}x^{A_i(t)}};
\]

3. if \( c = q \), then
\[
v_{x_q} = \frac{x^{A_j}w}{u_{x_q}x^{A_j(q-1)}}.
\]

Note that
\[
S(u_\pi f_{ij}, u_\sigma f_{il}) = LCM(u_\pi, u_\sigma)\left(\frac{x^{A_i}x^{B_j}x^{B_l}(u_\pi f_{ij})}{u_\pi x^{A_i}x^{B_j}} - \frac{x^{A_i}x^{B_j}x^{B_l}(u_\sigma f_{il})}{u_\sigma x^{A_i}x^{B_l}}\right),
\]

which simplifies to
\[
S(u_\pi f_{ij}, u_\sigma f_{il}) = LCM(u_\pi, u_\sigma)[x^{B_l}f_{ij} - x^{B_j}f_{il}] = -LCM(u_\pi, u_\sigma)x^{B_l}f_{jl} = w_{fjl}.
\]

So we need to show that
\[
w_{fjl} = \frac{w x^{A_i}}{x^{l(j(1))}}f_{j(j(1))} + \sum_{c=2}^{q-1} \frac{w x^{A_j} x^{A_i}}{x^{A_j(e-1)} x^{A_i(t)}} \frac{f_{j(e-1)j} f_{j(c)}}{x^{A_j(t)}} + \frac{w x^{A_j}}{x^{A_j(q-1)}} f_{j(q-1)l}
\]
is a standard expression of \( w_{fjl} \) with remainder zero. In other words we need to prove,
\[
w(x^{A_j}x^{B_l} - x^{A_l}x^{B_j}) = \frac{w x^{A_j} x^{B_l}}{x^{l(j(1))}} (x^{A_j} x^{B_l} - x^{A_j} x^{B_l})
\]

\[
+ \sum_{c=2}^{q-1} \frac{w x^{A_j} x^{A_i}}{x^{A_j(e-1)} x^{A_i(t)}} \left( x^{A_j(e-1)} x^{B_l} - x^{A_j(e-1)} x^{B_l(e-1)} \right) \quad [*]
\]

\[
+ \frac{w x^{A_j}}{x^{A_j(q-1)}} \left( x^{A_j(q-1)} x^{B_l} - x^{A_l} x^{B_l(q-1)} \right).
\]

Then [*] could be written as,
\[
w(x^{A_j}x^{B_l} - x^{A_l}x^{B_j}) = w(x^{A_j}x^{A_i} \frac{B_l}{x^{j(l)}} - x^{A_l}x^{B_j})
\]

\[
+ w x^{A_j} x^{A_l} (\sum_{c=2}^{q-1} \frac{B_l}{x^{A_j(c)}} - \frac{B_l}{x^{A_j(c-1)}})
\]

\[
+ w \left( x^{A_j} x^{B_l} - x^{A_j} x^{B_l(q-1)} \right).
\]
Rearranging the above we can see that equality holds. Now we will show that this is a standard expression with remainder zero. Let us note that \( x^{A_i}x^{B_l} > x^{A_j}x^{A_l} \) if \( i < j < l \) and also \( x^{A_i} > x^{B_j} \) for any \( i, j \). With these facts, we can see that

\[
w x^{A_j} x^{B_l} > \frac{w}{x^{A_{j(t(q-1))}}} x^{A_j} x^{A_l} x^{B_{j(t(q-1))}}
\]

because \( w \) has a \( x^{A_{j(t(q-1))}} \) term and \( j_{t(q-1)} < l \) but in the right-hand side \( x^{A_{j(t(q-1))}} \) is replaced by \( x^{B_{j(t(q-1))}} \). By this fact, we have a decreasing chain of monomials,

\[
w x^{A_j} x^{B_l} > \frac{w}{x^{A_{j(t(q-1))}}} x^{A_j} x^{A_l} x^{B_{j(t(q-1))}} > \cdots > \frac{w}{x^{A_{j(t(2))}}} x^{A_j} x^{A_l} x^{B_{j(t(2))}} > \frac{w}{x^{A_{j(t(1))}}} x^{A_j} x^{A_l} x^{B_{j(t(1))}}.
\]

Notice that this chain can also be written as

\[
w x^{A_j} x^{B_l} = \frac{wx^{A_j}}{x^{A_{j(t(q-1))}}} x^{A_{j(t(q-1))}} x^{B_l} > \frac{wx^{A_j}x^{A_l}}{x^{A_{j(t(q-2))}}x^{A_{j(t(q-1))}}} x^{A_{j(t(q-2))}} x^{B_{j(t(q-1))}} > \cdots > \frac{wx^{A_l}x^{A_j}}{x^{A_{j(t(1))}}x^{A_{j(t(2))}}} x^{A_{j(t(1))}} x^{B_{j(t(2))}} > \frac{wx^{A_l}}{x^{A_{j(t(1))}}} x^{A_j} x^{B_{j(t(1))}}.
\]

The above chain is the chain of leading terms of \([*]\) for each \( c \). So this with \([*]\) gives us a division algorithm that will lead \([*]\) to have a remainder zero when we divide by \( G \). With all these, we have shown that \( G \) is a Grobner basis of \( I(G) \).

\[ \square \]

**Theorem 14.** Let \( G \) be a simple graph on \([k]\), and let \( I_{n,k}(G) \) be the binomial tensor ideal of the polynomial ring \( R_{n,k} \). Then \( I_{n,k}(G) \) is a radical ideal.

**Proof.** By Theorem 13, \( I_{n,k}(G) \) has a Grobner basis of square-free binomials. Then by the proof of [9, Corollary 2.2], \( I(G) \) is radical. \[ \square \]
**Definition 38.** Let $G$ be a simple graph on $[k]$. Let $T \Gamma_{n,k}$ be a generic tensor on
\[ \Gamma_{n,k} = [2] \times [2] \times [2] \times \cdots \times [k]. \]
Then for each subset $S \subseteq [k]$ and for each element $l \in L(S)$ we define a prime ideal $P_{S_l}$ as follows. Let $T = [k] \setminus S$, and let $G_1, G_2, \ldots, G_{c(S)}$ be the connected component of $G_T$. Here $G_T$ is the restriction of $G$ to $T$ whose edges are exactly those edges $\{i, j\}$ of $G$ for which $i, j \in T$. For each $G_i$ we denote by $\bar{G}_i$ the complete graph on the vertex set $V(G_i)$. Now we set
\[ P_{S_l}(G) = \{l \cup \{I_{n,V(G_1)}(\bar{G}_1), \ldots, I_{n,V(G_{c(S)})}(\bar{G}_{c(S)})\}\}, \]
where $V(G_i)$ is the vertices of $G_i$.

We claim that the minimal primes of $I_{n,k}(G)$ are a subset of the above set of prime ideals. More precisely, there are subsets $S$ which give us minimal primes.

**Theorem 15.** Let $G$ be a simple graph on $[k]$. Then
\[ I_{n,k}(G) = \bigcap_{S \subseteq [k]} \bigcap_{l \in L(S)} P_{S_l}. \]
Before proving this theorem let us state and prove some lemmas. From now on we will use $I_G = I_{n,k}(G)$.

**Lemma 12.** Let $G$ be a simple connected graph on $k$ vertices, and let $P$ be a minimal prime of $I_G$. Then $\forall i \in [k]$, $P$ contains a variable from $x_{A_i}$ if and only if $P$ contains a variable from $x_{B_i}$.

**Proof.** We will use induction on $k$. The base case is when $k = 3$. The possible connected graphs on 3 vertices are a 3-path and a complete graph on three vertices. Since a complete
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graph on 3 vertices is in itself prime we are done. But when we have a 3-path graph we already discussed this in the previous section.

Now let us assume that for all \( k' \leq k - 1 \), the statement is true. Let \( P \) be a minimal prime of \( I_G \), and let \( A \) be a set of variables such that \( A \) contains exactly one variable from \( x_{A_i} \) for each \( i \leq k \). Let \( T \) be a maximal subset of \( A \) such that

- \( T \subset P \) and,

- \( T \) contains an element from \( x_{A_i} \) if and only if \( P \) does not contain any element from \( x_{B_i} \).

We will show that \( T \) must be empty. This tells us that a minimal prime ideal \( P \) of \( I_G \) contains an element from \( x_{A_i} \) if and only if it contains an element from \( x_{B_i} \).

Let us suppose \( T \) contains elements from the tensor \( T_i \) for all \( i \). Then we will show that \( P \) cannot be a minimal prime of \( I_G \). By Observation 3.1, for all edges \( \{i, j\} \) such that \( i < j \), the binomial corresponding to \( T_{i,j} \) is contained in \( T \). So \( P \supset T \supset I_{\tilde{G}} \). Where \( \tilde{G} \) is a complete graph on the vertex set of \( G \). But this is a contradiction to the fact that \( P \) is a minimal prime. Because \( I_{\tilde{G}} \) is a minimal prime of \( I_G \). This is due to the fact that \( I_{\tilde{G}} \) is a toric ideal. Corresponding to the lattice generated by the exponents of the binomial. So \( I_{\tilde{G}} \) must be a minimal prime of \( I \).

Next, let us suppose that \( T \neq \emptyset \) and \( T \) does not contain variables from \( T_i \) for all \( i \). So there is a \( j \) such that \( T \) does not contain any variable from \( T_j \). Then since \( G \) is connected, WLOG we can assume that there is an \( i \) such that \( \{i, j\} \) is an edge of \( G \) and \( T \) contains a variable
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from $T_i$. Thus the binomial of $T_{i,j}$ must be in $I \subset P$. So

$$\prod_{r \in A_i} x_r \prod_{s \in B_j} x_s - \prod_{s \in B_i} x_s \prod_{r \in A_j} x_r$$

is in $P$. By the property of $P$, we know that $\prod_{r \in A_i} x_r \prod_{s \in B_j} x_s \in P$ because $T$ contains a variable from $x_{A_i}$. Thus $\prod_{s \in B_i} x_s \prod_{r \in A_j} x_r \in P$, but $P$ does not contain any variable $x_{B_i}$ by its choice. So $\prod_{r \in A_j} x_r \in P$ and since $P$ is prime, $P$ should contain at least one variable from $x_{A_i}$. Since $T$ does not contain any variable from $x_{A_i}$, it follows that $P$ contains at least one variable from $x_{B_i}$.

Let $G'$ be the restriction of $G$ to the vertex set to $[k] \setminus j$. Let $x_{A_i}(P) \subset x_{A_i}$ be the set of variables in $P$ which are in $T_i$ and $x_{B_i}(P) \subset x_{B_i}$ be the set of variables in $P$ which are in $x_{B_i}$. Then

$$(I_{G'}, x_{a_j}(P), x_{b_j}(P)) = (I_G, x_{a_j}(P), x_{b_j}(P)) \subset P.$$ 

Thus $P = P \setminus \{x_{A_j}, x_{B_j}\}$ is a minimal prime ideal of $I'_G$ with the property that an element of $x_{A_i}$ is in $P$ but no element of $x_{B_i}$ is in $P$ for all $i$ such that $x_{a_i} \in T \subset P$. By induction hypothesis, $P$ is a minimal prime such that it contains a variable from $x_{A_i}$ if and only if it contains a variable from $x_{B_i}$. This contradicts the fact that $T \neq \emptyset$. So $T$ must be empty. 

Lemma 13. Let $G$ be a simple connected graph on $k$ vertices, and let $P$ be a minimal prime of $I(G)$. Then if $P$ contains a variable from $x_{A_i}$ for some $i$ then it contains exactly one variable from $x_{A_i}$ and one variable from $x_{B_i}$.

Proof. Let us suppose that $P$ contains more than one variable from $x_{A_i}$ for some $i$. First let $P'$ be an ideal which is contained in $P$ such that $P' = P \setminus \{\text{ one element from } x_{A_i} \cap P\}.$
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Since $P'$ is still a prime ideal that contains $I_G$ this contradicts the minimality of $P$. Hence if a minimal prime contains a variable from $x_{A_i}$ then it contains no more from $x_{A_i}$ and a similar argument works for $x_{B_i}$ too. Therefore if $P$ is a minimal prime of $I$ which contains a variable from $x_{A_i}$ then it contains exactly on variable from $x_{A_i}$ and by Lemma 12 $P$ contains exactly one variable from $x_{B_i}$.

**Lemma 14.** Let $G$ be a graph and let $G_1, G_2, \ldots, G_c$ be the connected components of $G$. Then let $P$ be a minimal prime of $I_G$ where $P$ has no variables in it. Then $P = (I_{G_1}, I_{G_2}, \ldots, I_{G_c})$.

**Proof.** Let $P$ be a minimal prime of $I_G$ with no variables in it. Then in order to prove the lemma, we claim that if $i, j$ with $i < j$ are two vertices of $G_k$ for some $k$, then the binomial of $T_{i,j}$ is in $P$, where $P$ is a minimal. From this, it will then follow that $(I_{G_1}, \ldots, I_{G_c}) \subset P$. Since $(I_{G_1}, \ldots, I_{G_c})$ is a prime ideal containing $I_G$, and $P$ is a minimal prime ideal containing $I_G$, we conclude that $P = (I_{G_1}, \ldots, I_{G_c})$.

Let $i = i_0, i_1, \ldots, i_r = j$ a path in $G_k$ from $i$ to $j$. We proceed by induction on $r$ to show that the binomial of $T_{i,j}$ is in $P$. The assertion is trivial for $r = 1$. Suppose now that $r > 1$. Our induction hypothesis says that the binomial of $T_{i_1,j}$ is in $P$. On the other hand, one has

$$\prod_{r \in A_{i_1}} x_r(\text{binomial of } T_{i,j}) = \prod_{r \in A_{i_1}} x_r(\text{binomial of } T_{i_i}) + \prod_{r \in A_i} x_r(\text{binomial of } T_{i_1,j}).$$

Thus $\prod_{r \in A_{i_1}} x_r(\text{binomial of } T_{i,j}) \in P$. Since $P$ is a prime ideal and since none of the variables from $x_{A_{i_1}}$ is in $P$, the binomial of $T_{i,j}$ must be in $P$. □

**Proof.** Theorem 15. It is clear that each of the sets in the right-hand side contains $I_G$. And
moreover, if $G$ is not connected let $G_1, \cdots, G_r$ be the connected components of $G$. Then each minimal prime $P$ of $I_G$ is of the form $P_1 + \cdots + P_r$ where $P_i$ is a minimal prime of $I_{G_i}$. Thus if each $P_i$ has the expected form stated in the theorem then so does $P$. Thus we can assume $G$ is connected. For each subset $S \subseteq [k]$, $L(S)$ is precisely the set such that it contains exactly one element from each $x_{A_i}$ and $x_{B_i}$ for $i \in S$. Then by Lemma 13 and Lemma 12 we can see that the variables in any minimal prime will be of the form stated in the theorem. From what we have shown, there exists a subset $S \subseteq [k]$ and an $l \in L(S)$ such that $P = \langle \{l, \tilde{P}\} \rangle$ where $\tilde{P}$ is a prime ideal containing no variables. Let $G'$ be the graph $G'[k] \setminus S$. Then the reduction modulo the ideal $\langle l \rangle$ shows that $\tilde{P}$ is a binomial prime ideal $I_{G'}$ which contains no variables. Let $G_1, \cdots, G_c$ be the connected components of $G'$. So if we show that $\tilde{P} = (I_{G_1} \cdots, I_{G_c})$, this then implies that $P = \langle \{l \cup \{\tilde{G}_1, \cdots, I(\Gamma_{n,k}, \tilde{G}_{c(S)})\}\} \rangle$, as desired. To simplify notation we may as well assume that $P$ contains no variables and have to show that $P = (I_{G_1} \cdots, I_{G_c})$, where $G_1, \cdots, G_c$ are the connected components of $G$. Then by Lemma 14 we have completed the proof.

Next, we will see, which of the subsets $S \subseteq [k]$ will be the minimal primes of $I_G$. In order to do this, first let us prove a proposition that will tell us for given subsets $S, T \subseteq [k]$ and $l \in L(S)$ and $l' \in L(T)$, when we have $P_{S_l} \subset P_{T_{l'}}$.

**Proposition 4.** Let $G$ be a simple graph on $[k]$ vertices, and let $S, T$ be subsets of $[k]$. Let $G_1, \cdots, G_s$ be the connected components of $G[k] \setminus S$ and let $H_1, \cdots, H_t$ be the connected components of $G[k] \setminus T$. Then $P_{T_{l'}} \subset P_{S_l}$ if and only if $T \subset S$ and $l' \subset l$ and for all $i = 1, \cdots, t$
one has $V(H_i) \setminus S \subset V(G_j)$ for some $j$.

Before proving this let us illustrate some examples to see why it should be true.

**Example 18.** Let $G$ be a 6-path and

$$M = \begin{pmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26}
\end{pmatrix}.$$  

Take $S = \{2, 4\}$ and $T = \{4\}$. Now, $C(G \setminus S) = \{1\}, \{3\}, \{5, 6\}$ and $C(G \setminus T) = \{1, 2, 3\}, \{5, 6\}$. From the previous chapter we know that $P_S$ and $P_T$ are minimal primes of $I_G$. So even though $T \subset S$, we have $P_T \not\subset P_S$. This is because $P_T$ contains the minor of the matrix \[
\begin{pmatrix}
  x_{11} & x_{13} \\
  x_{21} & x_{23}
\end{pmatrix},
\]
but $P_S$ does not contain this minor. However, $P_S$ contains all the variables in column 2 but $P_T$ does not. So neither of them contains the other. The key observation is that 1 and 3 are in different connected components of $G \setminus S$ but $\{1, 3\}$ lie in the same connected component of $G \setminus T$. To generalize this, let us consider the connected component $\{1, 2, 3\}$ of $G \setminus T$ and look at the set $\{1, 2, 3\} \setminus S = \{1, 3\}$. If this is not inside a connected component of $G \setminus S$ then as mentioned already the minor of 1, 3 is not in $P_S$ but it is in $P_T$. Thus if $T \subset S$ and $P_T \subset P_S$ then for each connected component $H$ of $G \setminus T$ there must exist a connected component $G_i$ of $G \setminus S$ such that $V(H) \setminus S \subset V(G_i)$.

Observation: For any subset $S, T \subset [k]$ and for any $l \in L(S)$ and $l' \in L(T)$, we have $P_{S_l} = (l, I_{G_1}, \cdots, I_{G_s})$ and $P_{T_{l'}} = (l', I_{H_1}, \cdots, I_{H_t})$. It then follows that $P_{S_l} \supset P_{T_{l'}}$ if and only if $l \supset l'$ and $S \supset T$ and $(l, I_{G_{t1}}, \cdots, I_{G_{ts}}) \supset (l', I_{H_{t1}}, \cdots, I_{H_{ts}})$. Note that, for the graph
\( H'_i = (H_i)_{[k]} \) for \( i = 1, \cdots, s \) we have
\[
(l, I_{\tilde{H}_1}, \cdots, I_{\tilde{H}_t}) = (l, I_{\tilde{H}'_1}, \cdots, I_{\tilde{H}'_t}).
\]
From this, it follows that \( P_{S_i} \supset P_{T'_l} \) if and only if \( l \supset l' \) and \( (l, I_{\tilde{G}_1}, \cdots, I_{\tilde{G}_s}) \supset (l, I_{\tilde{H}'_1}, \cdots, I_{\tilde{H}'_t}) \)
and this can happen if and only if \( l \supset l' \) and \((I_{\tilde{G}_1}, \cdots, I_{\tilde{G}_s}) \supset (I_{\tilde{H}'_1}, \cdots, I_{\tilde{H}'_t})\).

**Lemma 15.** Let \( Y_1, \cdots, Y_s \) and \( Z_1, \cdots, Z_t \) be pairwise disjoint subsets of \([k]\). Then
\[
(I_{Y_1}, \cdots, I_{Y_s}) \supset (I_{Z_1}, \cdots, I_{Z_t})
\]
if and only if for each \( i = 1, \cdots, t \) there is a \( j \) such that \( Y_j \supset Z_i \).

**Proof.** The converse direction is trivial. Before proving the forward direction, we need to develop some tools. Since we know that \( |A_1| = |B_1| \), we can fix a bijection \( \phi : B_i \to A_i \) between them. For each \( i \in [k] \) define a map \( \phi_i : B_i \to A_i \), where \( \phi_1 = \phi \) and for \( i \neq 1 \), \( \phi_i((b_1, \cdots, b_n)) = (\phi(b_{1i}), \phi(b_{12}), \cdots, i). \) Each \( \phi_i \) is a bijection. Now let us consider the \( K \)-algebra homomorphism
\[
\epsilon : K[T_{n,k}] \to K[x_{A_1}, \cup_{a \in A_1} x_a z_1] : \cdots, \{x_{A_s}, \cup_{a \in A_s} x_a z_s] \}_{i \in Y_s} \subset K[x_{T_{n,k}}, z_1, \cdots, z_s]
\]
given by \( \epsilon(x_{a_i}) = a_i \) for all \( a_i \in A_i \) for all \( i \) and \( \epsilon(x_{b_i}) = z_j(x_{\phi_i(b_i)}) \) for all \( b_i \in B_i \) and for all \( i \) and for \( i \in Y_j \) for \( j = 1, \cdots, s \).

Now coming back to the proof, let us assume that \( (I_{\tilde{Y}_1}, \cdots, I_{\tilde{Y}_s}) \supset (I_{\tilde{Z}_1}, \cdots, I_{\tilde{Z}_t}) \). Without loss of generality we can assume that \( \cup_{j=1}^t B_j = [k] \). Then \( \ker(\epsilon) \supset (I_{\tilde{Y}_1}, \cdots, I_{\tilde{Y}_s}) \). Now let us fix one of the sets \( Z_i \) and let \( m \in Z_i \). Then \( m \in Y_j \) for some \( j \). We claim that \( Z_i \subset Y_j \).
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If not, then let $v \in Z_i$ with $v < m$ and suppose $v \in Y_r$ for some $r \neq j$. Then

$$\prod_{r \in A_v} x_{A_v} \prod_{s \in B_m} x_{B_m} - \prod_{r \in A_m} x_{A_m} \prod_{s \in B_v} x_{B_v} \in I_{\tilde{Z}_i}.$$  

Then it follows that $\prod_{r \in A_v} x_{A_v} \prod_{s \in B_m} x_{B_m} - \prod_{r \in A_m} x_{A_m} \prod_{s \in B_v} x_{B_v} \in \ker(\epsilon)$.

So $\epsilon(\prod_{r \in A_v} x_{A_v} \prod_{s \in B_m} x_{B_m} - \prod_{r \in A_m} x_{A_m} \prod_{s \in B_v} x_{B_v}) = 0$, which implies $x_{A_v} x_{A_m} z_m^{2n-2} = x_{A_m} x_{A_v} z_v^{2n-2}$, which is a contradiction. \hfill \Box

**Proof.** Proposition 4. The proof directly follows from the Lemma 15 and the above observation. \hfill \Box

**Corollary 3.** Let $G$ be a connected simple graph on $[k]$ and $S \subset [k]$ and let $l \in L(S)$. Then $P_S$ is a minimal prime of $I_G$ if and only if $S = \emptyset$ and for each $i \in S$ one has $C(S \setminus \{i\}) < c(S)$.

**Proof.** Let us assume that $P_S(G)$ is a minimal prime ideal of $J_G$. Let $G_1, \cdots, G_r$ be the connected components of $G_{[k]\setminus S}$, and let $i \in S$. We have several cases,

- Suppose that there is no edge $\{i, j\}$ of $G$ such that $j \in G_m$ for some $m$. Set $T = S \setminus \{i\}$. Then the connected components of $G_{[k]\setminus T}$ are $G_1, \cdots, G_r, \{i\} = G_{r+1}$. Thus $C(T) = C(S) + 1$. However by Proposition 4 this is not possible because $T \subset S$ and for any $l \in L(S)$ there is a $l' \in L(T)$ such that $l' \subset l$ and for all $i = 1, \cdots, r + 1$ we have $V(G_i) \setminus S \subset V(G_i)$.

- Suppose there is exactly one $G_k$ such that $j \in G_k$ and $\{i, j\}$ is an edge of $G$. Then set again $T = S \setminus \{i\}$. Without loss of generality we can set $k = 1$. Then the connected
components of $T$ are $G'_1, \ldots, G_r$, where $G'_1 = G_1 \cup \{i\}$, thus $c(S) = c(T)$. Again by Proposition 4, this is not possible. Because $T \subset S$ and for any $l \in L(S)$ there is a $l' \in L(T)$ such that $l' \subset l$. And $V(G'_1) \setminus S = V(G_1) \subseteq V(G_1)$. So $P_{T'} \subset P_{S_i}$.

- Suppose there are at least two components $G_1, \ldots, G_d$, where $d \geq 2$, and $j_m \in G_m$ for $m = 1, \ldots, d$ such that $\{i, j_m\}$ is an edge of $G$. Then the connected components of $T = S \setminus \{i\}$ are $G_d, G_{d+1}, \ldots, G_r$, so $c(T) < C(S)$.

To prove the converse suppose, $c(S \setminus \{i\}) < c(S)$ for all $i \in S$. We want to show that for $l \in L(S)$, $P_{S_i}$ is a minimal prime of $I_G$. If it is not then for a fixed $S \subset [k]$ and for any $l, l' \in L(S)$ we cannot have $l \subset l'$. So there must exist a $T \subset S$ and $l' \in L(T)$ and $l \in L(S)$, where $l' \subset l$ such that $P_{T'}(G) \subset P_{S_i}(G)$. Let us pick an $i \in S \setminus T$. We have $C(S) \setminus \{i\} < C(S)$. Then by the above cases we considered we can assume that $G'_1, G_{d+1}, \ldots, G_r$ are the components of $G_{[k \setminus (S \setminus \{i\})]}$ where $V(G'_1) = \bigcup_{m=1}^{d} V(G_l) \cup \{i\}$ and $d \geq 2$. Then by the Proposition 4, there is a $H$ which is a connected component of $G_{[k \setminus T]}$ which contains $G'_1$. Then $V(H) \setminus S$ contains subsets $V(G_1)$ and $V(G_2)$. Hence $V(H) \setminus S$ is not contained in any of the $V(G_i)$. This is a contradiction to the fact that $P_{T'} \subset P_{S_i}$.
Chapter 4

Minimal primes of $3 \times 3$ adjacent minors of matrices

In the previous chapters, we discussed the 2-minors of matrices and tensors. In this chapter, we will discuss $3 \times 3$ adjacent minors of matrices. Work has been done to characterize the minimal primes of these ideals. Hosten and Sullivant in [11] studied $I_{m,n}(m)$, which denotes the adjacent $m \times m$ minors of an $m \times n$ matrix, where $n \geq m$. There they introduce prime sequences and prove that these correspond to the minimal primes of $I_{m,n}(m)$. In [3], Mohammadi et al. studied the general $I_\Delta(m)$ for any hypergraph $\Delta$ (which we will define shortly). There they define prime collections which also correspond to the minimal primes. In this chapter, we consider the hypergraph $G = \{123, 234, \ldots, (n-2)(n-1)n\}$ and show that the prime collections and prime sequences are the same. In addition, we give a promising technique to compute and characterize the minimal primes of the $3 \times 3$ adjacent
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minors of a 4 × n matrix. We illustrate this technique using an example. Finally, we propose a conjecture on the minimal primes of these ideals.

4.1 Adjacent minors of a 3 × n matrix

Adjacent minor ideals

Based on [11], we define the 3 × 3 adjacent minor ideal of a 3 × n matrix.

Definition 39. Let M be a 3 × n matrix with its entries as variables from the polynomial ring $K[x_{11}, x_{12}, \ldots, x_{3n}]$. The ideal $I_{3,n}(3)$ is the ideal generated by the determinants of 3 × 3 minors corresponding to the columns $\{123, 234, 345, \ldots, (n - 2)(n - 1)n\}$ of M.

Example 19. For $n = 5$ we have

$$M = \begin{pmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35}
\end{pmatrix}.$$ 

The 3 × 3 adjacent minor ideal of M is the ideal generated by the 3 × 3 minors of the columns $\{123, 234, 345\}$ of M:

$$I_{3,5}(3) = \langle \det \begin{pmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{pmatrix}, \det \begin{pmatrix}
  x_{12} & x_{13} & x_{14} \\
  x_{22} & x_{23} & x_{24} \\
  x_{32} & x_{33} & x_{34}
\end{pmatrix}, \det \begin{pmatrix}
  x_{13} & x_{14} & x_{15} \\
  x_{23} & x_{24} & x_{25} \\
  x_{33} & x_{34} & x_{35}
\end{pmatrix} \rangle.$$
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\[ I_{3,5}(3) = \langle x_{11}(x_{22}x_{33} - x_{32}x_{23}) - x_{12}(x_{21}x_{33} - x_{31}x_{23}) + x_{13}(x_{21}x_{32} - x_{31}x_{22}) \\
 x_{12}(x_{23}x_{34} - x_{33}x_{24}) - x_{13}(x_{22}x_{34} - x_{32}x_{24}) + x_{14}(x_{22}x_{33} - x_{32}x_{23}) \\
 x_{13}(x_{24}x_{35} - x_{34}x_{25}) - x_{14}(x_{23}x_{35} - x_{33}x_{25}) + x_{15}(x_{23}x_{34} - x_{33}x_{24}) \rangle. \]

Given any $n$, Hosten and Sullivant[11] have described the minimal primes of $I_{3,n}(3)$ using a notion called prime sequence.

**Definition 40.** Let $\Psi = \{[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]\}$, where $0 \leq a_i, b_i \leq n + 1$, be a sequence of $k$ intervals. The sequence $\Psi$ is called a **prime sequence** if it satisfies the following properties:

1. $\bigcup_{i=1}^{k}[a_i, b_i] = [0, n + 1]$,

2. $a_i < a_{i+1}, b_i < b_{i+1}$ for all $i$,

3. $b_i - a_i \geq 3$ for all $i$, and

4. $0 \leq b_i - a_{i+1} < 2$ for all $i$.

The definition says that each interval of $\Psi$ is a block of more than 3 consecutive columns and all together they cover all the columns of $X_{3n}$, including the two phantom columns 0 and $n + 1$. Moreover, the consecutive intervals in the sequence have a nonempty overlap of width less than 3. Given a prime sequence $\Psi$, we let $P_\Psi$ be the ideal in $K[x_{ij}]$ defined by

1. all $3 \times 3$ minors of $X[a_i, b_i]$ for each $[a_i, b_i] \in \Psi$. 

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2. all (maximal) \((b_i - a_{i+1} + 1) \times (b_i - a_{i+1} + 1)\) minors of \(X[a_i + 1, b_i]\) for \(1 \leq i \leq k - 1\).

In other words, \(P\) is generated by the 3-minors of the submatrices whose columns are indexed by the intervals in \(\Psi\), and the maximal minors of the submatrices whose columns are indexed by the overlap of consecutive intervals.

Example 20. Let \(R = K[x_{11}, x_{12}, \ldots, x_{16}, x_{21}, \ldots, x_{35}, x_{36}]\) and

\[
M = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36}
\end{pmatrix}
\]

Then the prime sequences are

\[
\Psi_1 = [0, 7]; \Psi_2 = [0, 3], [3, 7]; \Psi_3 = [0, 3], [2, 7]; \Psi_4 = [0, 4], [4, 7];
\]

\[
\Psi_5 = [0, 4], [3, 7]; \Psi_6 = [0, 5], [4, 7]; \Psi_7 = [0, 3], [2, 5], [4, 7].
\]

Theorem 16. [11] For a 3 × \(n\) matrix, \(P\) is a minimal prime of \(I_{3,n}(3)\) if and only if \(P = P_\Psi\) for a prime sequence \(\Psi\).

This is a strong theorem, which tells us that in order to find a minimal prime of \(I_{3,n}(3)\) it is sufficient to find a prime sequence. Mohammadi et al in [3] introduced another description of the same ideal in terms of hypergraphs. There they define what is called a prime collection. We will define these two things below and show that there is an isomorphism between prime collections and prime sequences. It is not intuitive that both the prime collection and the prime sequences are the same. So we explicitly translate the definitions and prove that they
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are the same. In the next section, we define the hypergraph ideals and their prime collections, and then later prove that each prime collection gives a prime sequence and vice-versa.

Example 21. Recall the previous example $I_{3,6}(3)$ and the prime sequences $\Psi_i$s. The corresponding minimal primes of $\Psi_1, \Psi_2, \Psi_3, \Psi_7$ are

- $P_{\Psi_1} = \langle 3 \times 3$ minors of columns of 3-element subsets of $\{1, 2, 3, 4, 5, 6\} \rangle$;
- $P_{\Psi_2} = \langle 3 \times 3$ minors of columns $\{1, 2, 3\}$ and 3-element subsets of $\{3, 4, 5, 6\}$, $x_{13}, x_{23}, x_{33} \rangle$;
- $P_{\Psi_3} = \langle 3 \times 3$ minors of columns $\{1, 2, 3\}$ and 3-element subsets of $\{2, 3, 4, 5, 6\}$ and all the $2 \times 2$ minors of columns $\{2, 3\} \rangle$;
- $P_{\Psi_7} = \langle 3 \times 3$ minors of columns $\{1, 2, 3\}, \{4, 5, 6\}$ and 3-element subsets of $\{2, 3, 4, 5\}$ and all the $2 \times 2$ minors of columns $\{2, 3\}$ and $\{4, 5\} \rangle$.

Hypergraph ideals

Let $K$ be a field, $m \leq n$ be two positive integers, $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates and $R = K[X]$ be the polynomial ring over $K$ in the indeterminates $x_{ij}$. It is often convenient to write determinants of submatrices of $X$ as $[I|J]_X$ where $I$ and $J$ are respectively the sets of rows and columns of the submatrix. If $I = [m]$, that is, the submatrix covers all rows of $X$, then we write $[J]$ for $[I|J]_X$. We denote by $x_i$ the $i^{th}$ column of $X$ and by $X_F$ the submatrix of $X$ with columns indexed by $F \subset [n]$. With this setting, let $G$ be a hypergraph on $[n]$ vertices, where all the edges of $G$ contain exactly $k$ vertices and $k \leq m$. We denote the edges of the hypergraph $G$ as $E(G)$. Then we can define the hypergraph
ideal to be the ideal generated by the $k \times k$ determinants of all submatrices $[I|J]_X$ of $X_{m \times n}$ where $J \in E(G)$ and $|I| = |J| = k$. Using concepts from matroid theory, Mohammadi et al. characterized the minimal primes of these hypergraph ideals in [3].

There are two special cases of the above problem. The first is when $k = m$. In this case we fix an $m \times n$ matrix and a hypergraph $G$ with each hyperedge containing exactly $m$ elements. Then we can say that the hypergraph ideal is generated by the $m \times m$ determinants of the submatrices $[I|J]_X$ of $X_{m \times n}$, where $J \in E(G)$ and $|I| = |J| = m$. Note here that since we are taking an $m \times m$ submatrix of an $m \times n$ matrix, for a fixed $J$ we have exactly one determinant.

An even more special case is when $k = m$ and we consider the hypergraph $G$ with hyperedges precisely the set of $m$ consecutive elements from $[n]$. If we define a hypergraph graph ideal for this case then it coincides with $I_{m,n}(m)$, the ideals studied by Hosten and Sullivant in [11]. As mentioned earlier, the minimal primes of this last case have been completely characterized in [11]. Finally, we restrict to the case where $m = 3$ and prove the equivalence between the characterization of the minimal primes in [11] and [3]. We restrict the more general definition of prime collection discussed in [3] to the hypergraph which has the edge set $\{123, 234, \ldots, (n-2)(n-1)n\}$ and carefully prove that the prime sequence discussed in [11] for $m = 3$ are the prime collections of this hypergraph ideals. At last we also prove that the ideals corresponding to the prime collections and prime sequence are identical.

**Definition 41** (Determinantal hypergraph ideal). A \textit{(simple) hypergraph} $\triangle$ on the vertex
set $[n]$ is a subset of the power set $2^{[n]}$. We assume that no proper subset of an element of $\Delta$ is in $\Delta$. The elements of $\Delta$ are called (hyper)edges. The determinantal hypergraph ideal of $\Delta$ is

$$I_\Delta = \langle [A|B]| : A \subseteq [m], B \in \Delta, |A| = |B| \rangle \subset R.$$

Note that when $m = 3$ and $\Delta = \{123, 234, \cdots, (n-2)(n-1)n\}$, $I_\Delta$ is exactly the $3 \times 3$ adjacent minor ideal of an $3 \times n$ matrix.

**Definition 42.** If $G$ is a graph that is an $n$-path

$$1 - 2 - 3 - 4 - 5 - \cdots - n,$$

the consecutive hypergraph of $G$ is

$$\Delta_G = \{123, 234, 345, \cdots, (n-2)(n-1)n\}.$$  

The definition of a consecutive hypergraph is more general [3] but we restricted to just the $n$-path graph and we directly gave the consecutive hypergraph of $G$. With this $\Delta_G$, $I_{\Delta_G}$ is exactly $I_{3,n}(3)$.

**Definition 43.** Let $G$ be the graph with the edge set $\{(1,2), (2,3), \cdots, (n-1,n)\}$. Then $S$ is a prime collection of singletons of $G$ if $S = \{s_1, s_2, \cdots, s_t\}$ where $s_1, s_2, \cdots, s_t \in [n]$, where $3 \leq s_i \leq n-2$ and $s_{i+1} \geq s_i + 3$.

**Example 22.** Let $G$ be an $n$-path and let $n = 10$. Then

1. $S = \{\{3\}, \{7\}\}$ is a prime collection of singletons of $G$. 

2. \( S = \{\{1\}, \{6\}, \{10\} \) is not a prime collection of singletons of \( G \) as \( \{1\}, \{n\} \in S \).

3. \( S = \{\{4,6\}, \{8\} \) is not even a set of singletons so not a prime collection of singletons of \( G \).

**Definition 44.** Let \( G \) be a graph on \( n \) vertices with the edge set
\[ E(G) = \{(1,2), (2,3), \ldots, (n-1,n)\}. \]
Let \( S \subseteq \mathcal{P}([n]) \) and \( S = \text{sing}(S) \cup S(2) \), where
\[ \text{sing}(S) = \{ \text{singletons in } S \} \] and \( S(2) \subseteq E(G) \setminus \{(1,2), \{n-1,n\}\}. \]
Let \( G' \) be the induced subgraph of \( G \) obtained by removing all vertices in \( \text{sing}(S) \) and \( S(2) \) and let \( C(S) \) be the set of connected components of \( G' \). We say \( PC(S) = S \cup C(S) \) is a prime collection of \( S \) for \( G \) if:

- \( \text{Sing}(S) \) is a prime collection of singletons of \( G \),

- \( \{m\} \in \text{Sing}(S) \) and \( \{l, l+1\} \in S(2) \) implies \( \{m-1, m, m+1\} \cap \{l, l+1\} = \emptyset \), and

- all the elements in \( S(2) \) are disjoint.

**Example 23.** Let \( G \) be the \( n \)-path.

1. If \( n = 10 \) and \( S = \{\{4\}, \{6,7\}, \{8,9\}\} \) then \( C(S) = \{\{1,2,3\}, \{5\}, \{10\}\} \). In addition, 
\[ \text{Sing}(S) = \{\{4\}\} \] and \( S(2) = \{\{6,7\}, \{8,9\}\} \). Therefore, 
\[ PC(S) = \{\{4\}, \{6,7\}, \{8,9\}, \{1,2,3\}, \{5\}, \{10\}\} \] is a prime collection.

2. If \( n = 12 \) and \( S = \{\{3\}, \{6\}, \{9,10\}\} \) then \( \text{Sing}(S) = \{\{3\}, \{6\}\} \), \( S(2) = \{\{9,10\}\} \), and \( C(S) = \{\{1,2\}, \{4,5\}, \{7,8\}, \{11,12\}\} \). Therefore, 
\[ PC(S) = \{\{3\}, \{6\}, \{9,10\}, \{1,2\}, \{4,5\}, \{7,8\}, \{11,12\}\} \] is a prime collection for \( G \).
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Observation:

- Let $G$ be the $n$-path, and for a subset $S \subseteq \mathcal{P}([n])$ let $PC(S)$ be a prime collection. We note that the elements of $Sing(S)$, $S(2)$, and $C(S)$ are disjoint and the union of their elements is $[n]$. Thus if $PC(S)$ is a prime collection we can construct a set $F_{PC(S)}$ such that,

$$F_{PC(S)} = \{f_1, f_2, \cdots, f_t\},$$

where $t = |sing(S)| + |S(2)| + |C(S)|$ and each $f_i$ consists of vertices and each $f_i$ is an element of either $Sing(S)$ or $S(2)$ or $C(S)$. We write $f_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{|f_i|}}\}$, such that $v_{i_1} < v_{i_2} < \cdots < v_{i_{|f_i|}}$. The set $F_{PC(S)}$ can be ordered such that the last element of $f_i$ is less than the first element of $f_{i+1}$.

- We know that if $f_i \in C(S)$ then $f_{i+1}, f_{i-1} \notin C(S)$, because otherwise $f_i \cup f_{i+1} \cup f_{i-1}$ would have been a connected component. Similarly, we know that if $f_i \in Sing(S)$ then $f_{i+1}, f_{i-1} \notin Sing(S)$ because $Sing(S)$ is a prime collection of singletons of $G$. However, it is possible that $f_i, f_{i+1}, f_{i-1} \in S(2)$.

- Now with the given $F_{PC(S)}$, we can construct another set $Q_{PC(S)}$ such that if $f_i \in C(S)$ then $f_i \cup f_{i+1} \cup f_{i-1}$ will be an element of $Q_{PC(S)}$. And if $f_i, f_{i+1} \in S(2)$ then $f_i \cup f_{i+1}$ will be an element of $Q_{PC(S)}$. Other than these two none of the other elements are in $Q_{PC(S)}$. Now we arrange $Q_{PC(S)} = \{q_1, q_2, \cdots, q_s\}$ such that all the $q_i$’s are ordered. Note that, $q_1 = f_1 \cup f_2$ and $q_s = f_t \cup f_{t-1}$, because $f_0, f_{t+1} = \emptyset$. Finally, note that the
first element of $q_{i+1}$ is greater than the first element of $q_i$. Similarly, the last element of $q_{i+1}$ is greater than the last element of $q_i$.

This is an important observation, which will help us later in the proof that prime collections are prime sequences. We will now illustrate this algorithm with an example.

**Example 24.** Let us recall the Example 23.

1. For $n = 10$ and $S = \{\{4\}, \{6, 7\}, \{8, 9\}\}$ we have

   \[
   PC(S) = \{\{4\}, \{6, 7\}, \{8, 9\}, \{1, 2, 3\}, \{5\}, \{10\}\}.
   \]

   Here, $Sing(S) = \{\{4\}\}$, $S(2) = \{\{6, 7\}, \{8, 9\}\}$, and $C(S) = \{\{1, 2, 3\}, \{5\}, \{10\}\}$. Thus,

   \[
   F_{PC(S)} = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7\}, \{8, 9\}, \{10\}\}.
   \]

   From this we construct $Q_{PC(S)}$: $\{1, 2, 3\} \in C(S)$, so $\{1, 2, 3\} \cup \{4\}$ is an element of $Q_{PC(S)}$. Similarly, $\{5\} \in C(S)$ implies $\{4\} \cup \{5\} \cup \{6, 7\} \in Q_{PC(S)}$. And $\{6, 7\}, \{8, 9\} \in S(2)$ implies $\{6, 7\} \cup \{8, 9\} \in Q_{PC(S)}$. So we have

   \[
   Q_{PC(S)} = \{\{1, 2, 3, 4\}, \{4, 5, 6, 7\}, \{6, 7, 8, 9\}, \{8, 9, 10\}\}.
   \]

2. For $n = 12$ and $S = \{\{3\}, \{6\}, \{9, 10\}\}$, we have $Sing(S) = \{\{3\}, \{6\}\}$, $S(2) = \{\{9, 10\}\}$, and thus $C(S) = \{\{1, 2\}, \{4, 5\}, \{7, 8\}, \{11, 12\}\}$. 

   Then we have $PC(S) = \{\{3\}, \{6\}, \{9, 10\}, \{1, 2\}, \{4, 5\}, \{7, 8\}, \{11, 12\}\}$. So

   \[
   F_{PC(S)} = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}.
   \]
And by using the algorithm described above we get

\[ Q_{PC(S)} = \{\{1,2,3\}, \{3,4,5,6\}, \{6,7,8,9,10\}, \{9,10,11,12\}\}. \]

**Definition 45.** Let \( G \) be an \( n \)-path, \( A \subset [n] \), \( B \subset \mathcal{P}([n]) \setminus \emptyset \). The set \( A \) is blocked by the set \( B \) if for any \( i < j \in A \) and for the unique path \( \pi : i, i+1, \cdots, j \), there exists an \( m \in B \) such that \( m \subset \{i+1, \cdots, j\} \).

**Example 25.** Let \( n = 10 \) and \( B = \{\{4\}, \{6,7\}, \{8,9\}\} \).

- If \( A = \{1,2,3,4\} \), then \( A \) is not blocked by \( B \).
- If \( A = \{6,7,8,9\} \), then \( A \) is not blocked by \( B \), because \( \{7,8\} \) does not contain any element of \( B \).
- If \( A = \{1,2,7\} \) then \( A \) is blocked by \( B \) because the path from 2 to 7 is 2, 3, 4, 5, 6, 7.
  
  This contains \( \{4\} \) which is an element of \( B \).

Next, we describe the minimal prime corresponding to a prime collection.

**Proposition 5.** [3] Let \( S \) be a subset of \( \mathcal{P}([n]) \) such that \( PC(S) \) is a prime collection. Then the ideal

\[ I_S = \langle \text{all the variables in the columns } \{m\} \text{ where } \{m\} \in \text{Sing}(S); \text{ all 2-minors of columns } \{l, l+1\} \text{ where } \{l, l+1\} \in S(2); \text{ 3-minors of columns of 3-subsets which are not blocked by } S \rangle \]

is a minimal prime of \( I_{\Delta_G} \). Moreover, all minimal primes are of this form.
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Example 26. Let $k = 10$ and $S = \{\{4\}, \{6, 7\}, \{8, 9\}\}$. We have seen

$PC(S) = \{\{4\}, \{6, 7\}, \{8, 9\}, \{1, 2, 3\}, \{5\}, \{10\}\}$, where $Sing(S) = \{\{4\}\}$, $S(2) = \{\{6, 7\}, \{8, 9\}\}$, and $C(S) = \{\{1, 2, 3\}, \{5\}, \{10\}\}$. Then

$I_S = \langle x_{14}, x_{24}, x_{34}, 2\text{-minors of } \begin{pmatrix} x_{16} & x_{17} \\ x_{26} & x_{37} \\ x_{36} & x_{37} \end{pmatrix}, 2\text{-minors of } \begin{pmatrix} x_{18} & x_{19} \\ x_{28} & x_{39} \\ x_{38} & x_{39} \end{pmatrix}, 3\text{-minor of } \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \rangle$.

Note here that the set $\{1, 2, 4\}$ is not blocked by $S$, and we need to include the 3-minors of the matrix $\begin{pmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{pmatrix}$, but we did not explicitly write it because all the variables in column 4 are in $I_S$ so the 3-minor is also in $I_S$.

Next, we prove the equivalence between a prime collection and a prime sequence.

Lemma 16. Given an $S \subset P(S)$ such that $PC(S)$ is a prime collection we get a prime sequence $\Psi_S$.

Proof. Let $Sing(S), S(2)$, and $C(S)$ denote the sets mentioned in Definition 44. By the Observation 4.1 we know that there is a $Q_{PC(S)}$ satisfying the mentioned properties. With this $Q_{PC(S)}$ we define a sequence $\Psi$ such that

$a_1 = 0,$

$a_s = n + 1,$
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\[ a_i = \text{first element of } q_i \text{ for } 1 < i \leq s, \]

\[ b_i = \text{last element of } q_i \text{ for } 1 \leq i < s. \]

Then we claim that \( \Psi = [a_1, b_1], [a_2, b_2], \ldots, [a_s, b_s] \) is a prime sequence.

1. \( \bigcup[a_i, b_i] = [0, n + 1] \), since \( Sing(S) \cup S(2) \cup C(S) \) is \([1, n] \) and since we did not omit any elements from this set in the construction of \( Q_{PC(S)} \) and we added \( 0, n + 1 \) in it.

This is true.

2. \( a_i < a_{i+1}, b_i < b_{i+1} \) for all \( i \) by the construction of \( Q_{PC(S)} \).

3. \( b_i - a_i \geq 3 \) for all \( i \): suppose \( q_i \) is of the type

\[ f_i \cup f_{i+1} \cup f_{i-1}, \]

where \( f_i \in C(S) \). Then if \( |f_i| > 3 \) we are done. If \( |f_i| \leq 3 \) (and we are assuming that \( f_{i-1}, f_{i+1} \) non-empty. But we will look at the case when one of them is empty later) then the possibilities are:

- \( |f_i| = 1 \). Then both \( f_{i+1}, f_{i-1} \) cannot be in \( Sing(S) \) because if \( f_j \in Sing(S) \) then \( f_{j+2}, f_{j-2} \) cannot be in \( Sing(S) \). And if \( f_{i+1} \in Sing(S) \), then \( f_{i-1} \in S(2) \), vice-versa so \( |f_i \cup f_{i+1} \cup f_{i-1}| = 4 \). Moreover, if both \( f_{i+1}, f_{i-1} \in S(2) \), then \( |f_i \cup f_{i+1} \cup f_{i-1}| = 5 \). So we are done.

- \( |f_i| = 2 \). Then the least possibility is that \( f_{i+1}, f_{i-1} \in Sing(S) \), this is possible because \( |f_i| = 2 \). Now \( |f_i \cup f_{i+1} \cup f_{i-1}| = 4 \), so we are done.
• $|f_i| = 3$, then since $f_{i+1}, f_{i-1}$ is non-empty, we have $|f_i \cup f_{i+1} \cup f_{i-1}| \geq 4$.

Now $f_{i-1} = \emptyset$ is only possible if $i = 1$. If we suppose $i = 1$ then we need to show that $|f_1 \cup f_2| \geq 4$. As we have seen, $q_1$ starts with zero. And if $|f_1| = 1$ then $f_2 \notin Sing(S)$ because in that case $f_2 = \{2\}$, and 2 cannot be in $Sing(S)$. Therefore $f_2 \in S(2)$, which implies $|f_i \cup f_{i+1}| = 4$. If $|f_1| = 2$ then $f_2$ can in $sing(S)$ or $S(2)$. In both cases we have $|f_i \cup f_{i+1}| = 4$. Finally, $|f_1| = 3$ readily gives us what we want. So we are done. Similarly, we can argue when $f_{i+1}$ is empty i.e. when $i = t$. Now let us consider the case when $q_i$ is of the type $f_j \cup f_{j+1}$ where $f_j, f_{j+1} \in S(2)$ then as we can see $|f_j \cup f_{j+1}| = 4$, we are done.

4. $0 \leq b_i - a_{i+1} < 2$ for all $i$:

• Suppose $[a_i, b_i]$ is of type $f_j \cup f_{j+1}$ and $[a_{i+1}, b_{i+1}]$ is also of a type $f_{j+1} \cup f_{j+2}$ such that $f_j, f_{j+1}, f_{j+2} \in S(2)$. Here since $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ are neighbors so should their corresponding elements of $P$. That is the reason we write $f_j, f_{j+1}, f_{j+2}$. So since $f_{j+1} \in S(2), b_i - a_{i+1} =$ second element of $f_{j+1}$—first element of $f_{j+1} = 1$.

• Suppose $[a_i, b_i]$ is of type $f_j \cup f_{j+1}$ and $[a_{i+1}, b_{i+1}]$ is of a type $f_{j+1} \cup f_{j+2} \cup f_{j+3}$. Then a similar argument shows that $b_i - a_{i+1} = 1$.

• Suppose $[a_i, b_i]$ is of type $f_{j-1} \cup f_j \cup f_{j+1}$ and $[a_{i+1}, b_{i+1}]$ is also of a type $f_{j+1} \cup f_{j+2} \cup f_{j+3}$. Then if $f_{j+1} \in Sing(S)$, then $b_i - a_{i+1} = 0$, or if $f_{j+1} \in S(2)$ then $b_i - a_{i+1} = 1$. 


Let us provide an example to illustrate the idea of the proof.

**Example 27.** 1. For \( k = 10 \) and \( S = \{\{4\}, \{6, 7\}, \{8, 9\}\} \), we had

\[
Q_{PC(S)} = \{\{1, 2, 3, 4\}, \{4, 5, 6, 7\}, \{6, 7, 8, 9\}, \{8, 9, 10\}\}.
\]

Thus we have the corresponding sequence \( \Psi_S = [0, 4][4, 7][6, 9][8, 11] \) is a prime sequence.

2. For \( k = 12 \) and \( S = \{\{3\}, \{6\}, \{9, 10\}\} \), we had

\[
Q_{PC(S)} = \{\{1, 2, 3\}, \{3, 4, 5, 6\}, \{6, 7, 8, 9, 10\}, \{9, 10, 11, 12\}\}.
\]

Then the corresponding sequence \( \Psi_S = [0, 3][3, 6][6, 10][9, 13] \) is a prime sequence.

**Lemma 17.** Given a prime sequence \( \Psi \) we get a \( S_\Psi \) such that \( PC(S_\Psi) \) is a prime collection.

**Proof.** Now we start with a prime sequence \( \Psi = [a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k] \), and we define the set \( \{b_i\} \in Sing(S) \) if \( b_i = a_{i+1} \) and \( \{(a_{i+1}, b_i)\} \in S(2) \) if \( b_i - a_{i+1} = 1 \). We claim that \( PC(S_\Psi) \) is a prime collection.

- First we need to show that \( Sing(S) \) is a prime collection of singletons for \( G \). This is true because, \( b_i \in Sing(S) \) if and only if \( b_i = a_{i+1} \). And since \( b_{i+1} - a_{i+1} \geq 3 \) the next singleton possible is \( b_{i+1} \) but since \( a_{i+1} = b_i \) we have that \( b_{i+1} - b_i \geq 3 \), this is possible for a prime collection of singleton of \( G \). So we are done.
• $S(2) \subset E(G) \setminus \{\{1, 2\}, \{n - 1, n\}\}$. First we show that $\{1, 2\} \notin S(2)$. Suppose
$\{1, 2\} \in S(2)$, then the first two elements in the prime sequence should be $[0, 2][1, m]$ for some $4 \leq m \leq n + 1$. But in this case, $b_1 - a_1 = 2$. So $\Psi$ is not a prime collection. Contradiction. A similar argument shows that $\{n - 1, n\} \notin S(2)$. If $\{a, b\} \in S(2)$ then $b - a = 1$ which implies $\{a, b\} = \{a, a + 1\} \subset E(G)$.

• Next we need to show that if $\{m\} \in Sing(S)$ and $\{l, l + 1\} \in S(2)$ then $\{m - 1, m, m + 1\} \cap \{l, l + 1\} = \emptyset$.

1. First, let us note that each element in $[n]$ can be in at most two blocks of a prime sequence. In particular, those two should be consecutive blocks. Thus if $b_i - a_{i + 1} = 0$ then $b_i \in [a_i, b_i] \cap [a_{i + 1}, b_{i + 1}]$ and hence there is no other element in the sequence which can contain $b_i$. So elements of $Sing(S)$ and $S(2)$ do not overlap. Therefore if $\{m\} \in Sing(S)$ and $\{l, l + 1\} \in S(2)$ then $\{m\} \cap \{l, l + 1\} = \emptyset$.

2. We now show that if $\{m\} \in Sing(S)$ and $\{l, l + 1\} \in S(2)$ then $\{m - 1\} \cap \{l, l + 1\} = \emptyset$. Suppose $m - 1 = l$. This is not possible by our argument for the previous case. Now suppose, $m - 1 = l + 1$. As a result we have consecutive blocks $[a_j, l + 1], [l, l + 1, m, b_{j + 1}]$ and another consecutive blocks $[a_i, m][m, b_{i + 1}]$. This is not possible as $m$ is in three different blocks of $\Psi$. A similar argument works to prove $\{m + 1\} \cap \{l, l + 1\} = \emptyset$.

• At last we need to show that the elements of $S(2)$ are disjoint. Suppose this is not the case, let $\{l, l + 1\} \in S(2)$. Then WLOG, there is a consecutive block $[a_i, l + 1][l, l +
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1, $l + 2, b_{i+1}$ and there is another consecutive block $[a_j, l + 2][l + 1, l + 2, b_{j+1}]$. This is not possible as $l + 2$ is in three different blocks of $\Psi$.

Example 28. 1. Let $n = 10$ and $\Psi = [0, 4][4, 7][6, 9][8, 11]$ be a prime sequence. Then $\text{Sing}(S) = \{\{4\}\}$ and $S(2) = \{\{6, 7\}, \{8, 9\}\}$. So $S = \{\{4\}, \{6, 7\}, \{8, 9\}\}$. From this we obtain $C(S) = \{\{1, 2, 3\}, \{5\}, \{10\}\}$ and

$PC(S) = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6, 7\}, \{8, 9\}, \{10\}\}$, which is a prime collection.

2. Let $n = 12$ and $\Psi = [0, 3][3, 6][6, 10][9, 13]$ be a prime sequence. Then $\text{Sing}(S) = \{\{3\}, \{6\}\}$ and $S(2) = \{\{9, 10\}\}$. So $S = \{\{3\}, \{6\}, \{9, 10\}\}$. From this we obtain $C(S) = \{\{1, 2\}, \{4, 5\}, \{7, 8\}, \{11, 12\}\}$ and

$PC(S) = \{\{1, 2\}, \{3\}, \{4, 5\}, \{6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$, which is a prime collection.

With Lemma 16 and Lemma 17 we have the theorem.

Theorem 17. There is a bijection between the set of prime collections and prime sequences.

Proof. Let $PC(S)$ be a prime collection corresponding to the subset $S$. Then by Lemma 16 we get a corresponding prime sequence $\Psi_S$. Now if we start with $\Psi_S$ with the algorithm used in Lemma 17 then we get the subset $S$ which corresponds to $PC(S)$ we started with. Thus we have a bijection between the prime collections and prime sequences.

Having shown the equivalence of the prime collection and prime sequence we should show that their corresponding minimal primes are also equivalent.
Lemma 18. Let $S \subset \mathcal{P}(\{n\})$ and let $\Psi_S = [a_1, b_1][a_2, b_2], \ldots, [a_s, b_s]$ be the corresponding prime sequence constructed using the procedure used in Lemma 16. Then a 3-subset \( \{x, y, z\} \subset [n] \) is not blocked by \( S \) if and only if \( \{x, y, z\} \subset [a_i, b_i] \) for some \( i \).

Proof. Suppose that \( \{x, y, z\} \subset [a_i, b_i] \) for some \( i \). Then suppose \([a_i, b_i]\) is of type, \( f_j \cup f_{j-1} \cup f_{j+1} \) for some \( j \), where \( f_j \in C(S) \). If \( \{x, y, z\} \subset f_j \) then it is clear that it is not blocked by \( S \).

We know that \( f_{j+1} \) and \( f_{j-1} \) could be in either \( S(2) \) or \( Sing(S) \). Suppose \( S \) blocks \( \{x, y, z\} \), then WLOG for the path \( \pi : x, x+1, \ldots, y-1, y \) we must have \( f_{j+1} \subset \{x+1, \ldots, y-1\} \) or \( f_{j-1} \subset \{x+1, \ldots, y-1\} \) but this is not possible because then either \( x \) or \( y \) is not in \([a_i, b_i]\).

Suppose \([a_i, b_i]\) is of type, \( f_j \cup f_{j+1} \) where both are in \( S(2) \). Then it is clear that for the path \( \pi : x, x+1, \ldots, y \) between any two elements \( x, y \in [a_i, b_i] \), the set elements excluding the endpoints is either \( \emptyset \) or one-element set or two-element set. In the first two cases, it is clear that the path is not blocked by \( S \). The third case is possible iff \( x = a_i \) and \( b_i = y \). However, the element \( \{a_i + 1, b_i - 1\} \notin S \). Hence it is not blocked by \( S \).

Conversely, suppose \( \{x, y, z\} \subset [n] \) is not blocked by \( S \). Suppose \( \forall i, \{x, y, z\} \notin [a_i, b_i] \). Then there is an \( i, j \) such that \( x \in [a_i, b_i] \setminus [a_j, b_j] \) and \( y \in [a_j, b_j] \setminus [a_i, b_i] \), WLOG \( i < j \). We can assume that \( j = i + 1 \) because in all the other cases the same argument will work.

- If \([a_i, b_i]\) is of type \( f_i \cup f_{i-1} \cup f_{i+1} \) where \( f_i \in C(S) \), then no matter what is \([a_{i+1}, b_{i+1}]\) we know that \( f_{i+1} \) is in it. By our choice \( x, y \notin f_{i+1} \). But the path \( x \) to \( y \) contains \( f_{i+1} \). Note that \( f_{i+1} \) is in either \( Sing(S) \) or \( S(2) \). So this is a contradiction.

- If \([a_i, b_i]\) is of type \( f_i \cup f_{i+1} \) where both \( f_i \) and \( f_{i+1} \) are in \( S(2) \). Then no matter what
[a_{i+1}, b_{i+1}] is we know that f_{i+1} is in it. By our choice x, y \notin f_{i+1}. By using the same argument as above we will arrive at a contradiction.

\[ \square \]

**Corollary 4.** Let G be an n-path, and let S be a set such that PC(S) is a prime collection on G and Ψ_S be the corresponding prime sequence. Then P_{Ψ_S} = I_S.

**Proof.** Let us start with an S such that PC(S) is a prime collection and construct a prime sequence Ψ_S according to the construction used in Lemma 16. Then we will show I_S = P_{Ψ_S}. The variables come from the columns m such that \{m\} \in Sing(S). By our construction if \{m\} \in Sing(S) then f_i = \{m\} for some i. And we know that f_{i+1}, f_{i-1} \in C(S), thus for some j we have [a_j, m][m, b_{j+1}] in the sequence Ψ_S. Hence variables in column m are also in P_{Ψ_S}. Similarly if \{m, m + 1\} \in S(2) then f_i = \{m, m + 1\} for some i. Then there are three possible cases:

- f_{i-1}, f_{i+1} \in C(S). Then we have for some j, [a_j, m + 1][m, b_{j+1}], so we are done.

- f_{i-1} \in C(S) and f_{i+1} \in S(2) or vice versa. In this case, f_{i-1} \cup f_{i-2} \cup f_i = q_j = [a_j, m+1] for some j and f_i \cup f_{i+1} = q_{j+1} = [m, b_{j+1}], so we are done.

- f_{i-1}, f_{i+1} \in S(2). In this case there is a j such that f_{i-1} \cup f_i = q_j and f_i \cup f_{i+1} = q_{j+1} so we have, a consecutive blocks [m - 2, m + 1][m, m + 4]. Thus the 2 × 2 minors of \{m, m + 1\} are in P_{Ψ_S}.

Then by Lemma 18, we have I_S = P_{Ψ_S}. \[ \square \]
4.2 Adjacent minors of $4 \times n$ matrices

After the minimal primes of a $3 \times 3$ adjacent minors of a $3 \times n$ matrix, the next question is whether we can compute the minimal primes of a $3 \times 3$ adjacent minor ideal of a $4 \times n$ matrix. This section is devoted to explaining an idea on how to solve this problem. Before getting into the theory let us look at an example. For this we take a $4 \times 4$ matrix. Let us denote the $3 \times 3$ adjacent minor ideal of the matrix,

$$M = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{pmatrix}$$

by $I_{4,4}(3)$. Our goal is to compute the minimal primes of $I_{4,4}(3)$.

Let $M_1 = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{pmatrix}$ and $M_2 = \begin{pmatrix}
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{pmatrix}$ and let $A_1$ and $A_2$ be the $3 \times 3$ adjacent minor ideal of $M_1$ and $M_2$, respectively. For convenience let us denote $A = I_{4,4}(3)$. Then it is clear that $A = A_1 + A_2$. Here

$$A_1 = \langle -x_{13}x_{22}x_{31} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{11}x_{22}x_{33},$$

$$-x_{23}x_{32}x_{41} + x_{22}x_{33}x_{41} + x_{23}x_{31}x_{42} - x_{21}x_{33}x_{42} - x_{22}x_{31}x_{43} + x_{21}x_{32}x_{43} \rangle$$

and

$$A_2 = -x_{14}x_{23}x_{32} + x_{13}x_{24}x_{32} + x_{14}x_{22}x_{33} - x_{12}x_{24}x_{33} - x_{13}x_{22}x_{34} + x_{12}x_{23}x_{34}.$$
CHAPTER 4. MINIMAL PRIMES OF $3 \times 3$ ADJACENT MINORS OF MATRICES

$$-x_{24}x_{33}x_{42} + x_{23}x_{34}x_{42} + x_{24}x_{32}x_{43} - x_{22}x_{34}x_{43} - x_{23}x_{32}x_{44} + x_{22}x_{33}x_{44}$$

Since we need minimal primes we look at the radical of $A$, which satisfies $\text{rad}(A) = \text{rad}(A_1 + A_2)$ (from now on we will denote the radical of an ideal $I$ as $\sqrt{I}$).

For any ideals $I, J$, we have that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$. So we rewrite the above equality as $\sqrt{A} = \sqrt{\sqrt{A_1} + \sqrt{A_2}}$. But we already have a characterization of the minimal primes of the two ideals $A_1$ and $A_2$.

For any $3 \times 4$ matrix there are two prime sequences $\Psi = \{ [0, 3], [2, 5] \}$ and $\Psi' = [0, 5]$. For $A_1$ the minimal prime corresponding to $\Psi$ is $I_1$ and for $\Psi'$ is $J_1$. Similarly for $A_2$ we have $I_2$ and $J_2$. Explicitly, $I_1 = \langle x_{23}x_{32} - x_{22}x_{33}, x_{23}x_{31} - x_{21}x_{33}, x_{22}x_{31} - x_{21}x_{32} \rangle$,

$J_1 = \langle \text{all 3-minors of } M_1 \rangle$, $I_2 = \langle x_{24}x_{33} - x_{23}x_{34}, x_{24}x_{32} - x_{22}x_{34}, x_{23}x_{32} - x_{22}x_{33} \rangle$, and $J_2 = \langle \text{all 3-minors of } M_2 \rangle$.

Hence $\sqrt{A_1} = I_1 \cap J_1$ and $\sqrt{A_2} = I_2 \cap J_2$. So we can write $\sqrt{A} = \sqrt{(I_1 \cap J_1) + (I_2 \cap J_2)}$

But

$$\sqrt{(I_1 \cap J_1) + (I_2 \cap J_2)} = \sqrt{(I_1 + J_1) \cap (I_2 + J_2)}$$

Now here are the interesting facts:

- The radical of the intersection is the intersection of radicals, so we have $\sqrt{(I_1 \cap J_1) + (I_2 \cap J_2)} = \sqrt{(I_1 + J_1)} \cap \sqrt{(I_2 + J_2)} \cap \sqrt{(I_2 + J_1)} \cap \sqrt{(I_1 + J_2)}$.

- $\sqrt{(I_1 + J_1)} \cap \sqrt{(I_2 + J_2)} \subset \sqrt{(I_2 + J_1)} \cap \sqrt{(I_1 + J_2)}$ (verified by Macaulay2 [8]). So we have

$$\sqrt{A} = \sqrt{(I_1 + J_1)} \cap \sqrt{(I_2 + J_2)}.$$
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Hence the minimal primes of $A$ comes from the minimal primes of RHS of the above equation. By computing in Macaulay-2 [8] the minimal primes of $\sqrt{(I_1 + J_1)}$ are $L_1 = \langle x_{33}, x_{32}, x_{23}, x_{22} \rangle$ and

$L_2 = \langle 2\text{-minors of } \begin{pmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{pmatrix} \rangle$. The minimal primes of $\sqrt{(I_2 + J_2)}$ are $L_3 = \langle 2\text{-minors of } \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \\ x_{32} & x_{33} \\ x_{42} & x_{43} \end{pmatrix} \rangle$ and $L_4 = \langle \text{all the 3 minors of } M \rangle$. Further, we verified that the minimal primes of $A$ are precisely $L_1, L_2, L_3, L_4$.

From the above case, we can see that for any $n$ we can look at the $4 \times n$ matrix as two $3 \times n$ matrices($M_1, M_2$) and by knowing the minimal primes of the adjacent minors ideals ($A_1, A_2$, respectively) we can determine the minimal primes of the former. By testing some lower dimensional cases we had a conjecture. Given two prime collections $S_1, S_2$ we call the corresponding minimal prime of $A_1$ as $I_{S_1}$ and $I_{S_2}$, respectively and similarly we call the $J_{S_1}$ and $J_{S_2}$ the corresponding minimal primes of $A_2$. We conjectured that

$$(I_{S_1} + J_{S_1}) \cap (I_{S_2} + J_{S_2}) \subset (I_{S_1} + J_{S_2}) \cap (I_{S_2} + J_{S_1}).$$

And this makes the components from the right-hand side redundant in the primary decomposition. If this conjecture is true then computing and studying the minimal primes of the ideals $I_S + J_S$ for each prime collection $S$ will lead to the minimal primes of the original problem. Unfortunately, this conjecture fails for $n = 6$ (verified by Macaulay-2 [8]). How-
ever, it is still worthwhile to study the minimal primes of $I_\Psi + J_\Phi$, where $\Psi$ and $\Phi$ are prime sequences.

At last, based on the work of [3] we state the following conjecture.

**Conjecture 1.** Let $M$ be $n \times 4$ matrix and let $G$ be a 4-path. Let $\triangle_G = \{123, 234\}$. Then the minimal primes of $I_{\triangle_G}$ are a subset of the minimal primes of $I_{4,n}(3)$.

As a first step towards proving the conjecture, we can think of the $n \times 4$ as a $4 \times n$ matrix. From the idea discussed above this $4 \times n$ matrix can be pictured as two $3 \times n$ matrices $M_1, M_2$. Let $A_1$ and $A_2$ be the $3 \times 3$ adjacent minor ideal of $M_1$ and $M_2$, respectively. Then $I_{4,n}(3) = A_1 + A_2$. Moreover, $\Psi = [0, n + 1]$ is a prime sequence of $[n]$. By Theorem 16 we know that $P_{1_\Psi}$ and $P_{2_\Psi}$ are minimal primes of $A_1$ and $A_2$, respectively. Recall that $P_{1_\Psi}$ and $P_{2_\Psi}$ are generated by all the $3 \times 3$ minors of $M_1$ and $M_2$, respectively. This implies that $P_{1_\Psi} + P_{2_\Psi} = I_{\triangle_G}$. Hence, if we could show that the minimal primes coming from $P_{1_\Psi} + P_{2_\Psi}$ are irredundant then we are done.
Bibliography


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