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THESIS TITLE: Repetition-free edge-colorings of k-ary trees

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Thesis Abstract

A repetition is a sequence of symbols in which the first half is the same as the second half. An edge-coloring of a graph is repetition-free if there is no path with a color pattern that is a repetition. The minimum number of colors so that a graph has an edge-coloring that is repetition-free is called the *Thue edge-chromatic number*. In this thesis we investigate the Thue edge-chromatic number of k-ary trees, that is trees in which every vertex has at most k children. Specifically we obtain new upper and lower bounds for the Thue edge-chromatic number of binary trees, and present a new general method for obtaining repetition-free edge-colorings of k-ary trees from what we call k-special sequences. We present examples of k-special sequences as well as algorithms for generating and verifying k-special sequences and repetition-free colorings of k-ary trees.

Keywords: edge-coloring, repetition-free, Thue edge-chromatic number, binary tree, k-ary tree

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Chapter 1

Introduction

1.1 Background

In Graph Theory it is natural to add colors to the edges or vertices of a graph. Though both types of colorings are interesting and have been studied extensively, in this thesis we will focus on edge-coloring. If we assign every edge the same color this is trivial and uninteresting so we introduce the rule that we do not want two edges with the same color to have a common endpoint. This inspires the question: following this rule what is the minimum number of colors to edge-color the graph? The answer to this is known as the edge-chromatic number of the graph. However, this is just one type of edge-coloring we can look at.

In 1906 Axel Thue [6, 7] proved that there are infinite sequences on three symbols that contain no repetition, such as abcabc. This theorem would later be used as the basis for repetition-free edge-colorings of graphs. In fact the minimum number of colors in a repetition-free edge-coloring is called the Thue edge-chromatic number. Unfortunately at the time Thue's theorem was published in a journal which had limited availability causing it to go relatively unnoticed according to Berstel [2]. Thue's work was later rediscovered, translated and became more widely known. As a direct result of Thue we know the Thue edge-chromatic number for paths. In this thesis we will study the Thue number for k-ary trees, an important class of trees which yields results for all trees.

1.2 Overview

We begin with a general introduction to Graph Theory focusing on basic definitions. No knowledge beyond what is covered in an introductory proof writing course should be necessary to read this thesis, but the material referenced in the bibliography can add to the understanding. In Chapter 2 we will cover what a repetition is and how it relates to edge-coloring. We include some quick results to familiarize the reader with repetition-free edge-colorings and state known lower and upper bounds for k-ary trees.

In Chapter 3 we use an exhaustive approach to improve upon the known lower bounds for the binary tree and explain why such an approach is not prudent past four colors. In Chapter 4 we give the main result of the thesis which is a proof of an upper bound for k-ary trees. We will transform our problem from a question about trees to a question about sequences with special properties. Lastly in the Appendix we will include data structures and pseudocode for useful programs that were used in the study of this problem.

1.3 Basic definitions

The following are standard definitions in Graph Theory accoring to West [9]. A **graph** $G = (V, E)$ consists of a set of **vertices** $V = V(G)$ and a set of **edges** $E = E(G)$. Each edge is an unordered pair of distinct vertices. We denote the number of vertices in a graph G, $|V(G)| = n$. Two vertices u, v are **adjacent** in G if they are contained in the same unordered pair in $E(G)$. If $u, v \in V(G)$ are adjacent, then we denote the edge containing both by uv and say that u, v are the **endpoints** of this edge. Our definition of a graph ensures that no vertex is adjacent to itself and that there is at most one edge uv for any choice of vertices u, v. The **degree** of a vertex v, denoted $d(v)$, is the number of edges containing it. The maximum degree of a vertex in G is denoted $\Delta(G)$. Thus for any graph G on n vertices $\Delta(G) \leq n - 1$.

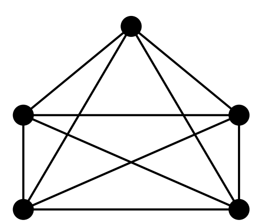

Figure 1.1: An example of a graph

An **isomorphism** from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say G is **isomorphic** H, written $G \cong H$, if there is an isomorphism from G to H.

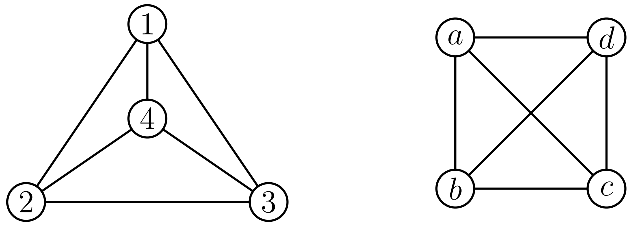

Figure 1.2: Example of isomorphic graphs

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of the endpoints to edges in H is the same as in G. We write $H \subseteq G$ and say that G **contains** H.

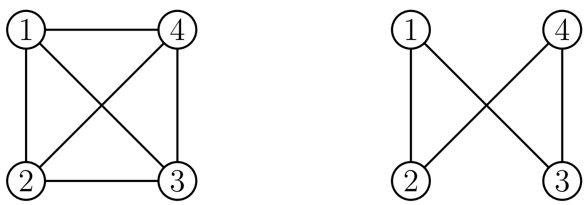

Figure 1.3: The graph on the right is a subgraph of the graph on the left

A **path** in a graph G is a sequence of distinct vertices $v_1v_2...v_j$ such that $v_iv_{i+1} \in E(G)$ for $1 \leq i \leq j-1$. The graph P_n is the path on n vertices with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. A graph G is called **connected** if there is a path between any two vertices of G. If there are two edge disjoint paths between two vertices of a graph G then we say that G contains a **cycle**. A graph G is called **acyclic** if it contains no cycles.

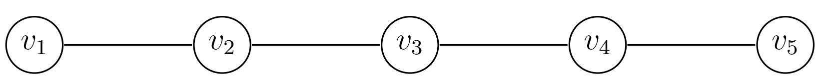

Figure 1.4: A drawing of P_5

1.3.1 Edge-coloring

A **m-edge-coloring** c of a graph G is a labeling of $E(G)$ with **colors** $\{c_1, c_2, \ldots, c_m\}$ that is a function $c : E(G) \to \{c_1, c_2, \ldots, c_m\}$. Usually we will assume that the m colors are the first m positive integers, that is $\{c_1, c_2, \ldots, c_m\} = \{1, 2, 3, \ldots, m\}$. We say a coloring is **proper** if distinct edges that have a common endpoint have different colors. The smallest m such that G has a proper m-edge-coloring is the **edge-chromatic number**, denoted $\chi'(G)$. Since edges containing the same vertex must receive distinct colors in a proper coloring we have $\chi'(G) \geq \Delta(G)$. Vizing [8] proved that this simple lower bound is almost optimal and that $\chi'(G) \leq \Delta(G) + 1$ for every graph G.

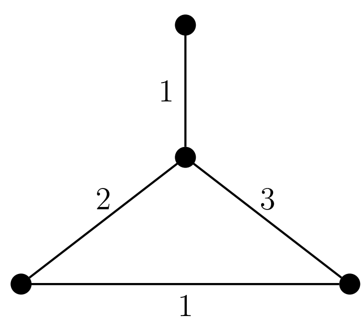

Figure 1.5: An example of a proper edge-coloring

1.3.2 Trees

A **tree** is a graph such that between any two vertices there is a unique path. From this definition we see that trees are acyclic and connected. Trees are the simplest type of connected graphs. As such, the study of trees is a good starting point for many graph parameters. For example it is not hard to see that $\chi'(G) = \Delta(G)$ when G is a tree. A **leaf** of a tree is a vertex of degree one.

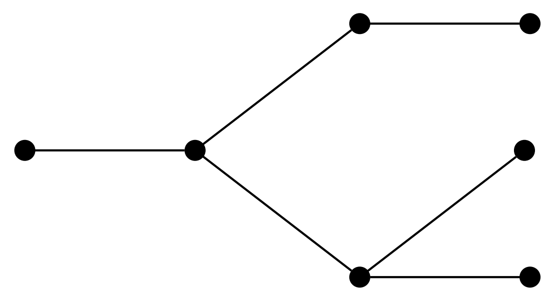

Figure 1.6: An example of a tree

1.3.3 k-ary trees

The **root** of a tree T is a distinguished vertex. As in computer science we will draw the root at the top and have the tree grow downwards from the root. A **k-ary tree** is a tree with maximum degree at most $k + 1$ and the root has degree at most k. In this thesis we will focus on k-ary trees. The **distance** between two vertices u, v in a connected graph, denoted $d(u, v)$, is the number of edges that belong on a shortest path between u and v. If $uv \in E(G)$ and $d(u, root) < d(v, root)$ then u is the **parent** of v, denoted $u = p(v)$, and v is a **child** of u. The **level** of a vertex is its distance to the root. The **height** of a tree is the maximum level. A **perfect k-ary tree** is a k-ary tree in which every vertex has either 0 or k children, and all leaves have the same level.

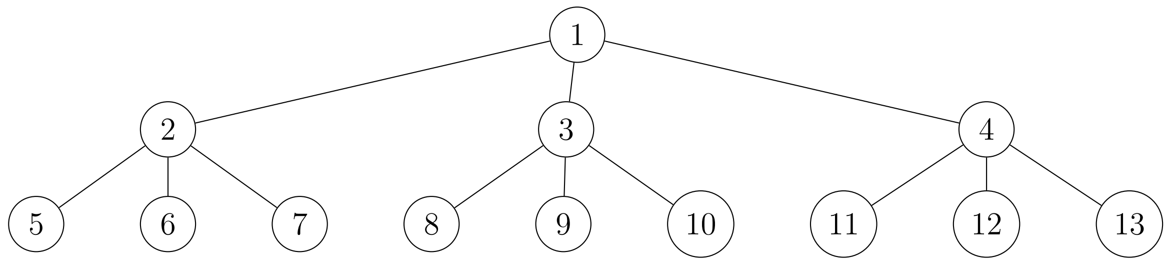

Figure 1.7: Perfect $T_{3,2}$ with root 1

The path on n vertices, P_n, can be viewed as a 1-ary tree, where the root is one of the leaves. A 2-ary tree is called a **binary tree**. We will focus on the binary tree in later chapters. $T_{k,h}$ is the perfect k-ary tree of height h. If $h \geq 2$ then $\Delta(T_{k,h}) = k + 1$. Every tree T is isomorphic to a subgraph of $T_{\Delta(T)-1,h}$ for some h, so if we can edge-color perfect k-ary trees it will generalize to all trees.

Chapter 2

Repetition-free sequences and edge-colorings

2.1 Repetitions and sequences

Let Σ be a set of symbols and let $S = s_1, s_2, \ldots, s_{2r}$ be a sequence with $s_i \in \Sigma$ and $r \geq 1$. A sequence S is a **repetition of order r** if $s_i = s_{i+r}$ for all $i = 1, 2, \ldots, r$. For example, the sequence 134134 is a repetition of order three. A **block** is a subsequence of consecutive terms in a given sequence. A sequence is **repetition-free** if it contains no block that is a repetition. The **length** of a sequence is the number of terms in the sequence.

Example 2.1.1. *The following sequence 1342312314421 contains a repetition highlighted in red while 121314 is repetition-free.*

The following theorem is crucial to this thesis.

Theorem 2.1.2 (Thue,1906 [6]). *Arbitrarily long repetition-free sequences can be created using only three symbols.*

The proof of this theorem is constructive, and uses simultaneous substitutions over a given set of symbols. For instance the substitution seen in the paper of Alon, Grytczuk, Haluszczak, and Riordan [1],

$$1 \to 12312$$

$$2 \to 131232$$

$$3 \to 1323132$$

preserves the repetition-free property. If we have a repetition-free sequence on symbols $\{1, 2, 3\}$ we can make it longer by replacing individual symbols with the blocks above. We have color coded the blocks in the example below to make the substitution more apparent.

Example 2.1.3. *We extend the repetition-free sequence 1231 to 12312131232132313212312.*

2.1.1 Palindromes

A **palindrome** is a sequence of symbols, $P = p_1, p_2, \ldots, p_n$ for $n \geq 2$ that looks the same when written backwards, that is $P = p_n, p_{n-1}, \ldots, p_1$. Observe that palindromes contain a shorter palindrome of length two (if n is even) or three (if n is odd) at their center. A sequence that is repetition-free may be a palindrome or contain palindromes as a block. A sequence that does not contain a palindrome as a block is called **palindrome-free**.

Example 2.1.4. *For instance, 13231 is a palindrome, repetition-free, and has the palindrome 323 at its center. Note that 321123 also is a palindrome* *with 11 at the center, which is a repetition.*

A repetition-free, palindrome-free sequence of arbitrary length can be created with four symbols. To create such a sequence start with a repetition-free sequence on three symbols. Then we remove any palindromes by inserting a fourth symbol after every block of two symbols as mentioned in the paper of Brešar, Grytczuk, Klavžar, Niwczyk, and Peterin [3].

Theorem 2.1.5. *There are arbitrarily long repetition-free palindrome-free sequences on four symbols.*

Example 2.1.6. *3213123 is repetition-free sequences that is a palindrome on symbols* $\{1, 2, 3\}$*. If we insert the symbol 4 as shown we get the following sequence, 3241341243 which is repetition-free and palindrome-free.*

Observe that this will be palindrome free since it can not contain a palindrome of length two or three. Let $P = p_1, p_2, \ldots, p_n$ be a palindrome and repetition-free with $n \geq 2$ with center palindrome P'. If n is even then P' is length two which contradicts the fact that P is repetition-free. If n is odd then P' is length three. If a fourth symbol is placed between every block of two symbols of P then the fourth symbol will also be placed in P' which will no longer be a palindrome.

Thue's theorem led to the study of repetition-free sequences and their use for coloring of graphs. Repetition-free colorings of both edges and vertices have been studied; however this thesis will focus on repetition-free edge-colorings.

2.2 Repetition-free sequences and graphs

2.2.1 Thue edge-chromatic number

Let G be a graph and for $v, v' \in V(G)$ we denote the color on edge vv' by $c(vv')$. The **color pattern** of a path $P = v_1, v_2, \ldots, v_j$ is the sequence $c(v_1v_2), c(v_2v_3), \ldots, c(v_{j-1}v_j)$. An edge-coloring of a graph G is repetition-free if no path in G has a repetition as a color pattern.

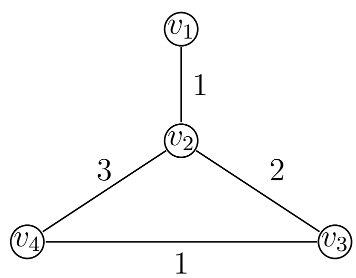

Figure 2.1: The path $P = v_1, v_2, v_3, v_4$ has color pattern 121

Observation 2.2.1. *The sequence 12311143 has a repetition of order 1, namely 11. If a path had such a color pattern then two edges with a common endpoint would share the same color, so the coloring would not be proper. In general, repetition-free edge-colorings must be proper.*

Thue's [6, 7] study of arbitrarily long repetition-free sequences was inspiration for the following terminology. The **Thue edge-chromatic number**, $\pi'(G)$, is the smallest number of colors in a repetition-free coloring of the edges of G. Since colorings must be proper it immediately follows that $\pi'(G) \geq \chi'(G)$ though in most cases $\pi'(G) > \chi'(G)$. This makes finding the Thue edge-chromatic number an interesting problem. Recall that $\chi'(G) \geq \Delta(G)$, so if we find a proper coloring that uses $\Delta(G)$ colors and is repetition-free then $\pi'(G) = \chi'(G) = \Delta(G)$. For the graph G in Figure 2.1 the coloring is repetition-free so we have $\pi'(G) = \chi'(G) = \Delta(G) = 3$.

Recall that $T_{k,h}$ is the perfect k-ary tree of height h. Another example where equality holds is the graph $T_{k,1}$, which is also known as a star graph, due to its appearance when the root is drawn in the center.

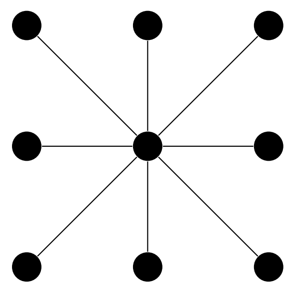

Figure 2.2: A star graph

Proposition 2.2.2. *$\pi'(T_{k,1}) = \chi'(T_{k,1}) = \Delta(T_{k,1}) = k$.*

Proof. The maximum degree of $T_{k,1}$ is k so it suffices to color the edges of $T_{k,1}$ using colors $1, 2, \ldots, k$. Any path in $T_{k,1}$ has at most two edges. This coloring is proper so edges sharing a common endpoint receive distinct colors, therefore no path contains a color pattern that is a repetition.

□

2.2.2 Thue edge-chromatic number of P_n

Recall that P_n is the path on n vertices. The path P_n can be properly edge colored by alternating two colors on its edges, and so $\chi'(P_n) = 2$. Since the Thue edge-chromatic number of a graph is based on the color pattern of paths it is important to know the Thue edge-chromatic number of P_n.

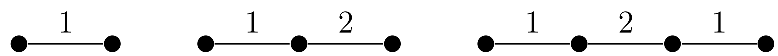

Figure 2.3: Repetition-free edge-colorings

In Figure 2.3 we can see proper repetition-free edge-colorings for P_2, P_3, and P_4. So we conclude that $\pi'(P_2) = 1$ and $\pi'(P_3) = \pi'(P_4) = 2$.

Theorem 2.2.3. *If* $n \geq 5$ *then* $\pi'(P_n) = 3$.

Proof. When $n \geq 5$ a proper 2-edge-coloring of P_n will contain the repetition 1212. So $\pi'(P_n) \geq 3$ for $n \geq 5$. Theorem 2.1.5 shows that there are arbitrarily long repetition-free sequences on three symbols which can be used as a color pattern for P_n, so $\pi'(P_n) \leq 3$. Thus $\pi'(P_n) = 3$ for $n \geq 5$. □

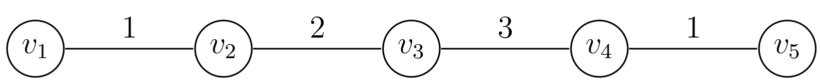

Figure 2.4: A repetition-free edge-coloring of P_5

If a graph G has a path on more than three vertices as a subgraph then $\pi'(G) \geq 3$, providing an important lower bound for connected graphs.

2.2.3 Thue edge-chromatic number of C_n

After studying paths the natural next step is to study cycles. C_n is the cycle on n vertices. The smallest cycle is C_3 and is shown in Figure 2.5. To properly edge color C_3 you need three colors. To properly edge color C_n where n is odd you need three colors.

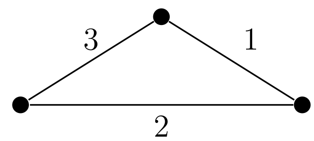

Figure 2.5: A repetition-free edge-coloring of C_3

When $n \geq 5$ the graph C_n will contain a path with four edges as a subgraph so $\pi'(C_n) \geq 3$.

Theorem 2.2.4. $\pi'(C_5) = 4$

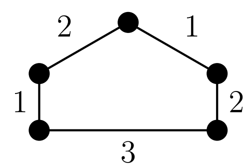

Figure 2.6: A proper edge-coloring of C_5

Proof. A proper coloring of C_5 requires three colors. By the pigeon hole principle there is a color that is used only once. Consider the path created by excluding the edge whose color is used only once (in Figure 2.6 it is color 3). This path has a color pattern that is a repetition. Replace any color on this path with a fourth color and the path will no longer have a repetition. Thus $\pi'(C_5) = 4$. □

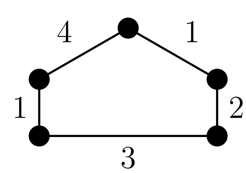

Figure 2.7: A repetition-free edge-coloring of C_5

By computing the Thue edge-chromatic number of C_n for $n \leq 2001$ Alon, Grytczuk, Haluszczak, and Riordan [1] found that $\pi'(C_i) = 4$ for $i = \{5, 6, 9, 10, 14, 17\}$ and $\pi'(C_i) = 3$ for all other $i < 2002$, which led them to conjecture that every other cycle has a Thue edge-chromatic number of three. This conjecture was later proved by Currie resulting in Theorem 2.2.5.

Theorem 2.2.5 (Currie [4]). $\pi'(C_n) = 3$ *for* $n \geq 18$

2.3 Repetition-free colorings of perfect k-ary trees

2.3.1 Labeling $T_{k,h}$

We will now discuss how to label $T_{k,h}$. This labeling will be used extensively in Chapter 3 and Chapter 4 of this thesis. Let $V(T_{k,h}) = \{1, 2, \ldots, n\}$ and $E(T_{k,h}) = \{e_1, e_2, \ldots, e_{n-1}\}$. The root receives label 1 and its children receive labels $2, 3, \ldots, k$ respectively from left to right. We label the vertices of level two $k+1, k+2, \ldots$ from left to right. Label the remaining levels in a similar fashion. For $1 \leq i \leq n-1$ let e_i be the edge with endpoints $i+1$ and $p(i+1)$. We call this a **labeled** k-ary tree.

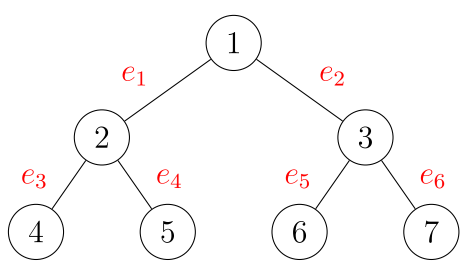

Figure 2.8: A labeled $T_{2,2}$

There are several benefits to such a labeling; we are able to quickly determine the parent of a vertex and we can quickly find the labels of the children of a vertex. The children of each non-leaf vertex $i \in V(T_{k,h})$ are $(i-1)k+2, (i-1)k+3, \ldots, (i-1)k+k+1$. Let $j \in V(T_{k,h})$ be any non-root vertex. The parent of i is $p(i) = i'$ where $i' = \lfloor (i+k-2)/k \rfloor$.

Observation 2.3.1. *Note that for the binary tree where $k = 2$ the labeling simplifies. The parent of a vertex i is $\lfloor i/2 \rfloor$ and the children of $i \in V(T_{2,h})$ are $(i-1)2+2 = 2i-2+2 = 2i$ and $(i-1)2+2+1 = 2i-2+2+1 = 2i+1$, as we can see in Figure 2.8.*

As a result this labeling is convenient for working with the binary tree.

Example 2.3.2. *Let $k = 3$ and $h \geq 3$. Observe that $(7-1)3+2 = 20$, $(7-1)3+3 = 21$, and $(7-1)3+3+1 = 22$, so vertex 7 has children $20, 21, 22$. Next $\lfloor (7+3-2)/3 \rfloor = \lfloor 8/3 \rfloor = 2$, so $p(7) = 2$.*

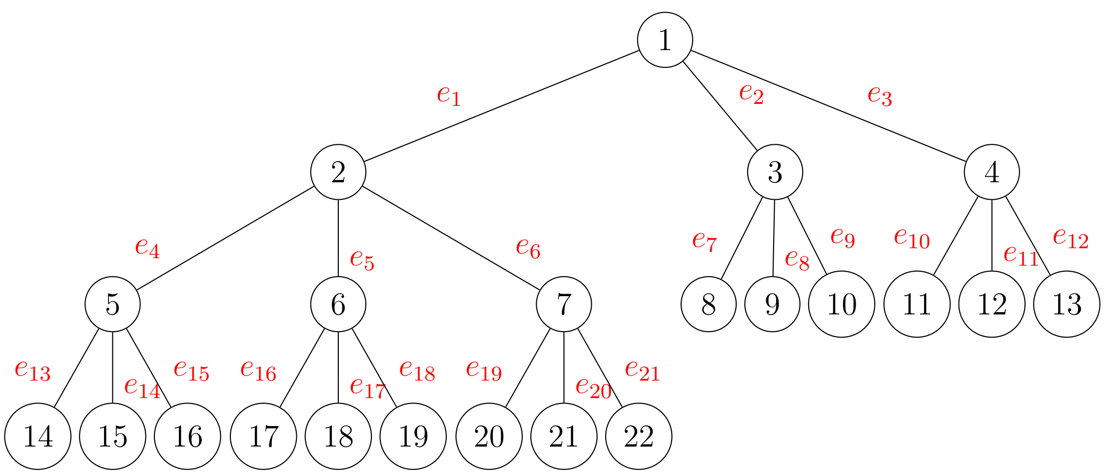

Figure 2.9: A subgraph of a labeled $T_{3,3}$

In $T_{k,h}$ we say a path $P = v_1, v_2, \ldots, v_j$ is **going up** if $v_1 > v_2 > \cdots > v_j$ and **going down** if $v_1 < v_2 < \cdots < v_j$. If there is an m with $1 < m < j$ such that $v_1 > v_2 > \cdots > v_m < \cdots < v_j$ then we call v_m the **turning point**. A path that is going down has no turning point and can only be extended by continuing to go down, that is, by adding children of the last vertex in the path. A path that is going up with no turning point is called **monotone**. If $P = v_1, v_2, \ldots, v_j$ is going down then the path $P' = v_j, v_{j-1}, \ldots, v_1$ is going up, as a result every path in $T_{k,h}$ can be viewed as either monotone or having exactly one turning point.

Example 2.3.3. *Consider the labeling of $T_{2,2}$ from Figure 2.8. The path $4, 2, 1$ is going up and is monotone. The path $1, 3, 6$ is going down while $6, 3, 1$ is going up. The path $3, 1, 2, 5$ has turning point 1.*

2.3.2 Known Thue edge-chromatic numbers for k-ary trees

Recall that P_n can be viewed as a 1-ary tree of height $n - 1$ where the root is a leaf, so for $h \geq 4$, $\pi'(T_{1,h}) = \pi'(P_{h+1}) = 3$ by Theorem 2.1.5 and for the star graph we have seen that $\pi'(T_{k,1}) = k$ in Proposition 2.2.2. This is a lower bound for perfect k-ary trees, though it is not a very good one. A much better lower bound for $T_{k,h}$ follows from the following result.

Theorem 2.3.4 (Sudeep,Vishwanathan [5])**.** *$\pi'(T_{k,2}) = \lfloor 1.5k \rfloor + 1$ and $\pi'(T_{k,3}) > 1.61k$.*

We also have an upper bound of $4k$ for the Thue edge-chromatic number for trees (Theorem 2.3.6) thanks to Alon, Grytzcuk, Haluszczak, and Riordan [1]. The following is based on their proof and has been adapted for perfect k-ary trees.

Definition 2.3.5. *For a natural number m, $Rem_k(m)$ is defined to be k if k divides m, and the remainder when dividing m by k otherwise.*

Theorem 2.3.6 (Alon, Grytzcuk, Haluszczak, Riordan [1])**.** *$\pi'(T_{k,h}) \leq 4k$ for all k, h and $\pi'(T_{k,h}) \leq 3k$ for $h \leq 5$.*

Proof. Label $T_{k,h}$ as in the previous section. We will show how to use a palindrome-free repetition-free sequence $B = b_1, b_2, \ldots, b_h$ on m symbols to obtain a repetition-free (mk)-coloring on m symbols. The result then follows from Theorem 2.1.5 and the observation that the sequence 12312 is repetition-free and palindrome-free.

The edges of $T_{k,h}$ are naturally partitioned by level. These edges form vertex disjoint stars that have k edges each.

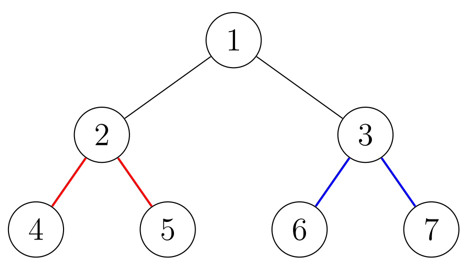

Figure 2.10: Two stars formed by the edges in red and blue

Let e_m be an edge of level j. Color the edges of level j so that $c(e_m) = (b_j, Rem_k(m))$. For each ordered pair there are four choices for the first coordinate and k possibilities for the second. By the multiplication principle we are using $4k$ colors. Consider vertex i which has children $ik - k + 2, \ldots, ik + 1$. The edges above the children of i are $e_{k(i-1)+1}$, $e_{k(i-1)+2}, \ldots, e_{ik}$ respectively. Since $Rem_k(k(i-1)+1), Rem_k(k(i-1)+2), \ldots, Rem_k(ik) = 1, 2, \ldots, k$ the colors on the children of i will be distinct. B

is repetition-free so $c(ip(i))$ is different in the first coordinate than the colors used on the children of i. So this is a proper coloring.

Suppose that this edge-coloring produces a repetition on the color pattern of some path $P = v_0, v_1, \ldots, v_{2r}$ of $T_{k,h}$. Our coloring is proper so the repetition must have order greater than 1 so $r \geq 2$ and P must have more than four vertices. Let $P' = p_1, p_2, \ldots, p_{2r}$ be the sequence of the first coordinates of the color pattern of P. Since P has a repetition on its color path, we observe that $p_i = p_{i+r}$ for $i = 1, 2, \ldots, r$.

Case 1: P is monotone.

By definition P is going up and has no turning point. In this case P' is a subsequence of B that contains a repetition which is a contradiction. So P is not monotone.

Case 2: P is not monotone.

Since P is not monotone it must have a turning point say at v_m for $m \in \{1, \ldots, 2r-1\}$. Let l be the level of v_m and l' be the level of v_{2r}.

Suppose the turning point is in the middle so that $v_m = v_r$. Notice that for $j = 1, 2, \ldots, r$, vertices v_j and v_{2r-j} are in the same level, so $p_j = p_{2r-j+1}$. Observe that $p_1 = p_{r+1} = p_r$, $p_2 = p_{r+2} = p_{r-1}$, $\ldots$, so $p_1, p_2, \ldots, p_r$ is a palindrome. The path $v_r, v_{r-1}, \ldots, v_0$ has sequence of first coordinates of its color pattern $p_r, p_{r-1}, \ldots, p_1$ which is a subsequence of B, a contradiction. Therefore the turning point of P can not be in the middle.

Suppose $v_m \neq v_r$. The edges on either side of v_m are consecutive edges belonging to the same star. These edges have colors (b_{l+1}, z) and (b_{l+1}, z') for some $z, z' \in \{1, 2, \ldots, k\}$. If $m < r$ the edges on either side of v_{m+r} have colors (b_{l+1}, z) and (b_{l+1}, z'), but are in consecutive levels of $T_{k,h}$, so B must contain the repetition b_{l+1}, b_{l+1}. Similarly if $m > r$, then the edges on either side of v_{m-r} share first coordinates but are in consecutive levels so B must contain the repetition b_{l+1}, b_{l+1}.

In either case we arrive at a contradiction so we deduce there is no path with a repetition for a color pattern. □

Example 2.3.7. *Here is an example of the coloring described in the proof for $T_{2,3}$ using B=124.*

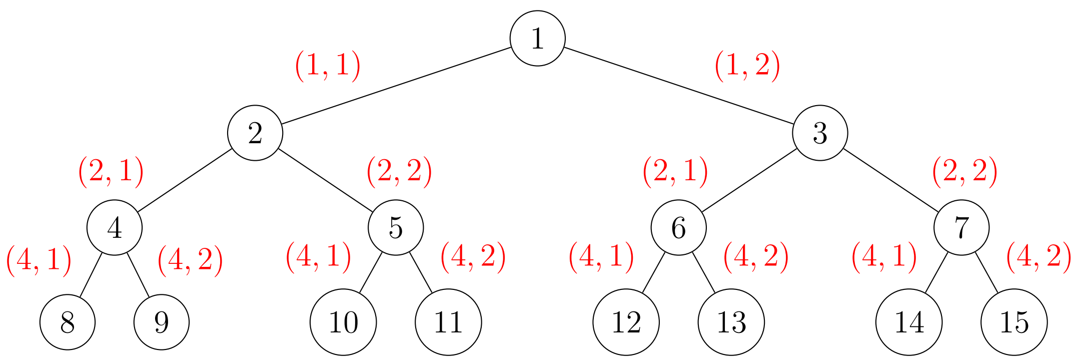

Figure 2.11

Table 2.1 is a summary of the results in this chapter for $T_{k,h}$. The columns represent k and the rows are the height h of the tree. If there is only one entry in a cell it means that the Thue edge-chromatic number is known. If there are two entries the value on the left is the lower bound and the value on the right is the upper bound.

k/h	1	2	3	4, 5	≥ 6
1	1	2	2	3	3
2	2	4	4, 6	4, 6	4, 8
3	3	5	5, 8	5, 9	5, 12
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	k	$\lfloor 1.5k \rfloor + 1$	$1.61k, \lfloor 2.5k + 1 \rfloor$	$1.61k, 3k$	$1.61k, 4k$

Table 2.1

The first column is Proposition 2.2.2 and the top row is Theorem 2.1.5. For the second column it is easy to see that by using k completely new colors on children of each vertex in level 2 we can extend the coloring showing that $\pi'(T_{2,k}) \leq 1.5k + 1$ to prove that $\pi'(T_{3,k}) \leq 2.5k + 1$. All of the lower bounds are from Theorem 2.3.4. All of the upper bounds are from Theorem 2.3.6.

Chapter 3

Exhaustive search for repetition-free colorings of k-ary trees with few colors

3.1 Introduction

Table 2.1 shows all the choices of k, h for which $T_{k,h}$ can be colored with 3 colors or fewer. Thus it seems interesting to study perfect k-ary trees that can be 4-colored. Since every $T_{k,h}$ with $k \geq 3$ and $h \geq 2$ requires at least five colors this effectively reduces our investigation to stars, paths, and binary trees. We already know the Thue edge-chromatic number for paths and stars from Chapter 2 so the focus will be on binary trees with the hope that results from binary trees may generalize to other k-ary trees with $k > 2$. The main theorem of this chapter will prove that binary trees requires at least five colors if $h > 3$.

Recall that in our labeling of $T_{k,h}$, e_i is the edge with endpoints $i + 1$

and $p(i+1)$.

Definition 3.1.1. *The* ***color sequence*** *of a labeled k-ary tree is a sequence $S = s_1, s_2, \ldots$ where $s_i = c(e_i)$ for $1 \leq i \leq n-1$. Consecutive terms of the color sequence are called a* ***color block****.*

It is natural to partition T_k by levels so it also feels natural to partition the color sequence similarly. A color block may contain part or all of the edge colors of a given level.

Example 3.1.2. *The color sequence of the graph in Figure 3.1 is: 12343412132412, where level two is shown in red and is a color block.*

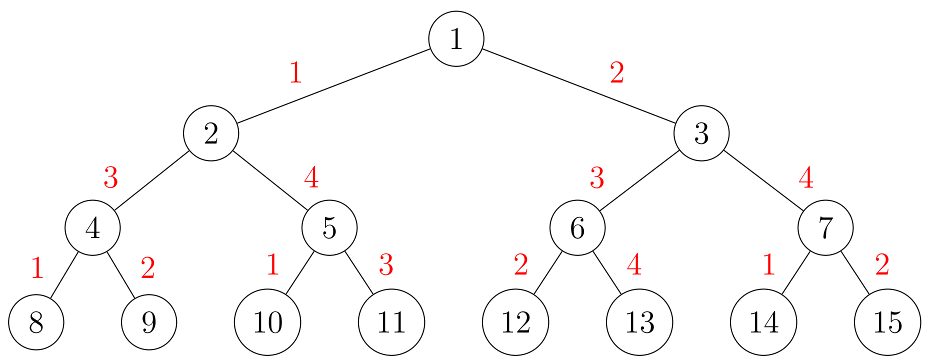

Figure 3.1

Definition 3.1.3. *We say two colorings c_1, c_2 of $T_{k,h}$ are* ***color isomorphic*** *if there is an isomorphism $f : V(T_{k,h}) \to V(T_{k,h})$ such that $c_1(e) = c_1(e')$ if and only if $c_2(f(e)) = c_2(f(e'))$. We say that f is a* ***color isomorphism****.*

Example 3.1.4. *Consider the colorings c_1, c_2 shown in Figure 3.2.*
$f := \{(1,1),(2,3),(3,2),(4,6),(5,7),(6,4),(7,5)\}$ is an isomorphism on $T_{2,2}$ with $c_2(f(e)) = c_1(e)$. Thus c_1, c_2 are isomorphic.

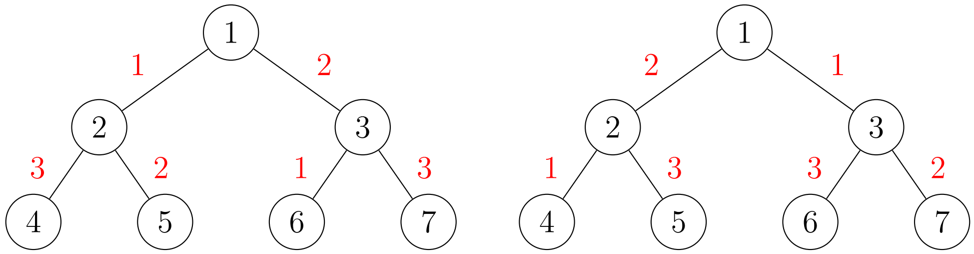

Figure 3.2: Isomorphic Colorings of $T_{2,2}$

Example 3.1.5. *The colorings in Figure 3.3 are color isomorphic with f being the identity function. Here $c_2(e) = Rem_3(c_1(e) - 1)$, that is*

Color

$1 \to 3$

$2 \to 1$

$3 \to 2$

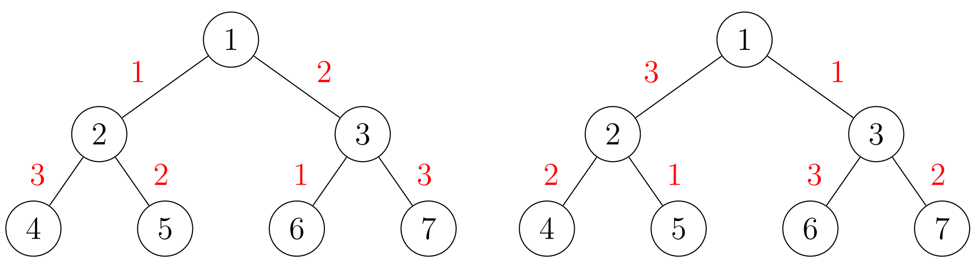

Figure 3.3: Isomorphic Colorings of $T_{2,2}$

Remark 3.1.6. *Essentially in 3.1.4 we only permute the vertices but keep the coloring fixed, whereas in 3.1.5 we only permute the colors but keep the vertices fixed. A general color isomorphism could permute both the vertices and the colors.*

Definition 3.1.7. *Let c be a repetition-free edge-coloring of a graph G that* *uses colors* $\{1, 2, \ldots, m\}$. *We say that a repetition-free coloring* c' *is an* ***extension*** *of* c *to a graph* H *if:*

1. G *is a subgraph of* H

2. $c(e) = c'(e)$ *for all* $e \in E(G)$

3. $c'(E(H)) \subseteq \{1, 2, \ldots, m\}$

We say that a repetition-free m*-coloring* c *on* $T_{k,h}$ *can not be* ***extended*** *if it has no extension to* $T_{k,h+1}$. *We say that color* j *is* ***forced*** *on an edge* $e \in E(H) - E(G)$ *if every extension* c' *of* c *to* H *has* $c'(e) = j$. *We say that an edge* $e \notin E(G)$ *is* ***uncolorable*** *if* c *can not be extended to* $G + e$.

Example 3.1.8. *Let us extend the repetition-free edge-coloring of Figure 3.4 using colors* $\{1, 2, 3, 4\}$.

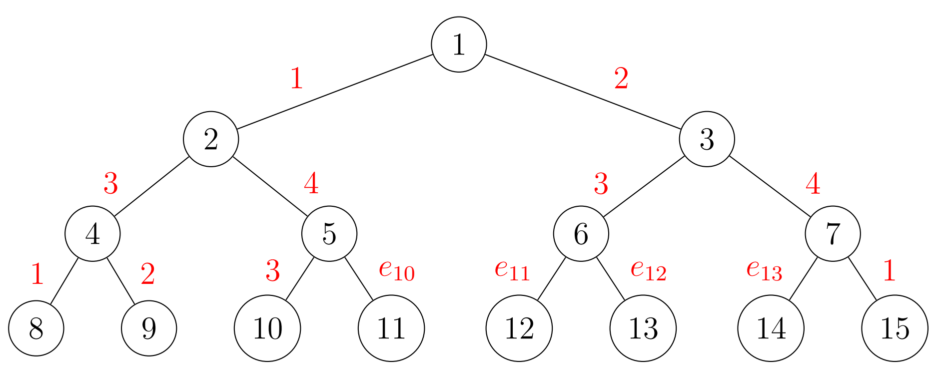

Figure 3.4: A repetition-free edge-coloring of a labeled $T_{2,3}$

Observe that $c(e_{10}) \neq 2$ *otherwise we have a repetition on path* $11, 5, 2, 1, 3, 7, 15$ *with color pattern* 241241. *Also we must exclude colors 4 and 3 on* e_{10} *to have a proper coloring. Thus* $c(e_{10})$ *is forced to be 1.*

We may finish this extension with the following colors: $c(e_{11}) = 2$, $c(e_{12}) = 4$, *and* $c(e_{13}) = 2$.

Definition 3.1.9. *Let i be a vertex in a labeled $T_{k,h}$. The* ***left child*** *of i is the child with label $(i-1)k+2$ and the* ***right child*** *of i is the child with label $(i-1)k+k+1$. The* ***left edge*** *is the edge with endpoints i and $(i-1)k+2$ and the* ***right edge*** *is the edge with endpoints i and $(i-1)k+k+1$.*

As mentioned previously this labeling is particularly nice for binary trees as vertex i of a labeled $T_{2,h}$ has left child $(i-1)2+2 = 2i$ and right child $(i-1)2+2+1 = 2i+1$.

We may say that the colors of the k edges between some vertex v and its children $T_{k,h}$ are forced if there are exactly k colors available since the order of these colors will not affect whether a repetition is present or not. In this case we usually extend a coloring by using the colors in increasing order from the left child to the right child.

Example 3.1.10. *If we flip the colors of the left edge and right edge in the last level the only difference will be the vertices at the start and end of the path with a repetition that is a color pattern, the internal vertices will be the same.*

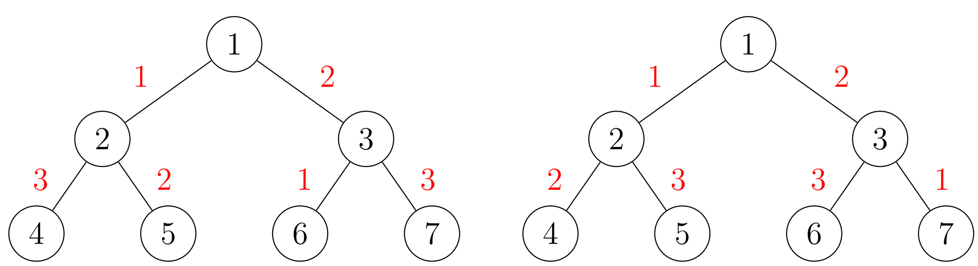

Figure 3.5: Let T be the graph on the left and T' the graph on the right

Observe that T has repetition $5, 2, 1, 3, 6$ with color pattern 2121 and T' has repetition $4, 2, 1, 3, 7$ with color pattern 2121.

Example 3.1.11. *Let us extend the repetition-free edge-coloring of Figure 3.6 using colors* $\{1, 2, 3, 4\}$.

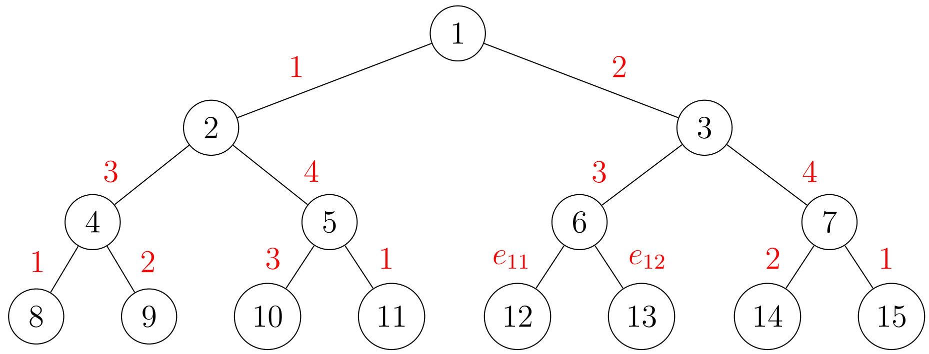

Figure 3.6: A repetition-free edge-coloring of a labeled $T_{2,3}$

We must exclude color 3 for $c(e_{11})$ *and* $c(e_{12})$ *to have a proper coloring. If* $c(e_{11})$ *or* $c(e_{12})$ *is color 1 then we have a repetition on the path* $9, 4, 2, 1, 3, 6$ *and last vertex 12 or 13 respectively, with color pattern* 231231. *Thus* $c(e_{11})$ *and* $c(e_{12})$ *are forced to be* $2, 4$ *in some order, and we choose* $c(e_{11}) = 2$ *and* $c(e_{12}) = 4$.

Remark 3.1.12. *Every coloring of* $T_{2,h}$ *is isomorphic to one where the color on the left child is smaller than to the color on the right child for every vertex.*

3.2 Binary tree

In the previous chapter we noted that Sudeep, Vishwanathan [5] proved the following bounds for binary trees, $\pi'(T_{2,2}) = \lfloor 1.5(2) \rfloor + 1 = 4$ and $\pi'(T_{2,3}) > 1.61(2) = 3.22$. In fact $\pi'(T_{2,3}) = 4$ because Figure 3.7 is a repetition-free coloring of $T_{2,3}$.

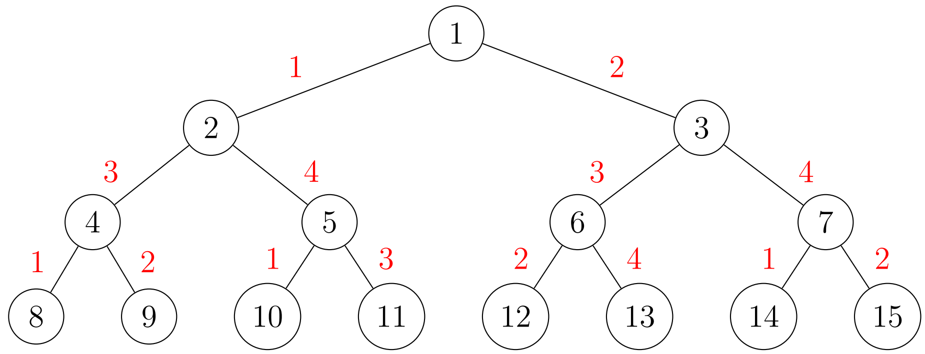

Figure 3.7: A repetition-free 4-coloring of $T_{2,3}$

We improve upon the lower bound using an exhaustive approach. In the next chapter we will discuss ideas for the upper bound. The main purpose of this section is to prove the following theorem.

Theorem 3.2.1. *$\pi'(T_{2,h}) = 4$ for $2 \leq h \leq 3$ and $\pi'(T_{2,h}) = 5$ for $4 \leq h \leq 10$.*

Theorem 3.2.1 has sharper lower and upper bounds than any of the theorems we have seen so far. The main challenge is to prove that $\pi'(T_{2,4}) > 4$. The proof is lengthy so we will break the proof into several steps by first determining up to isomorphism all repetition-free 4-colorings of $T_{2,h}$ for $h \leq 3$.

3.2.1 Repetition-free 4-colorings of $T_{2,1}$

Lemma 3.2.2. *Every repetition-free 4-coloring of $T_{2,1}$ is isomorphic to the one in Figure 3.8.*

Proof. Observe that in any repetition-free edge-coloring of $T_{2,1}$ that $c(e_1) \neq c(e_2)$. So the identity function is a color isomorphism between c and the coloring in Figure 3.8. So we may assume that $c(e_1) = 1$ and $c(e_2) = 2$. □

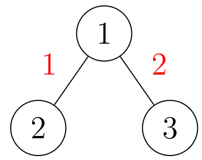

Figure 3.8: $\pi'(T_{2,1}) = 2$

Without loss of generality we can now use Figure 3.8 as a starting point for finding all non-isomorphic colorings of $T_{2,2}$, which we will do in the next section.

3.2.2 Repetition-free 4-colorings of $T_{2,2}$

Lemma 3.2.3. *Every repetition-free 4-coloring of $T_{2,2}$ is color isomorphic to either Figure 3.9 or Figure 3.10.*

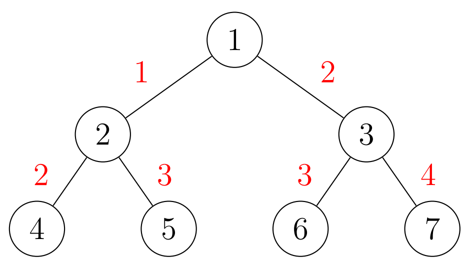

Figure 3.9

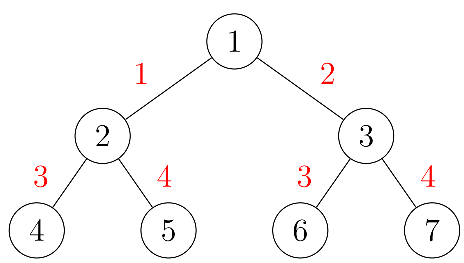

Figure 3.10

Proof. It can be checked by case analysis that the colorings in Figure 3.9 and Figure 3.10 are repetition-free. From Sudeep, Vishwanathan[5] we know that $\pi'(T_{2,2}) = 4$ so since we start by extending Figure 3.8, color 3 and color 4 must each be used at least once on an edge in level two. Since this is the first time either color 3 or color 4 has been used we can assign them without loss of generality as they will be isomorphic.

Case 1:

If we use neither color 1 nor color 2 on the edges level two then vertex 3 and vertex 4 must both be incident to color 3 and color 4. Thus we arrive at the graph from Figure 3.10.

Case 2: Suppose that color 2 appears on the edges of level two, that is $c(e_3) = 2$ or $c(e_4) = 2$.

We may assume that $c(e_3) = 2$, then without loss of generality $c(e_4) = 3$ and we can extend this coloring uniquely to the one in Figure 3.9, since using color 1 on e_5 would yield a repetition on the color pattern of 4,2,1,3,6.

Case 3: Suppose that color 1 appears on edges of level two, that is $c(e_5) = 1$ or $c(e_6) = 1$.

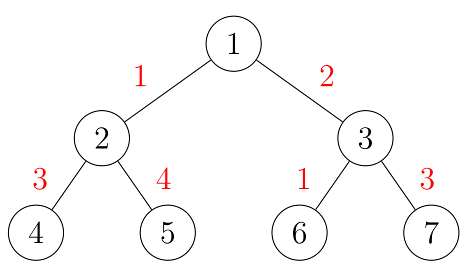

Figure 3.11

We may assume that $c(e_5) = 1$, then without loss of generality $c(e_6) = 3$ and we can extend this coloring uniquely to the one in Figure 3.11.

However Figure 3.9 and Figure 3.11 are isomorphic (see Figure 3.12) with isomorphism,

$$f : \{(1,1),(2,3),(3,2),(6,4),(4,6),(5,7),(7,5)\}.$$

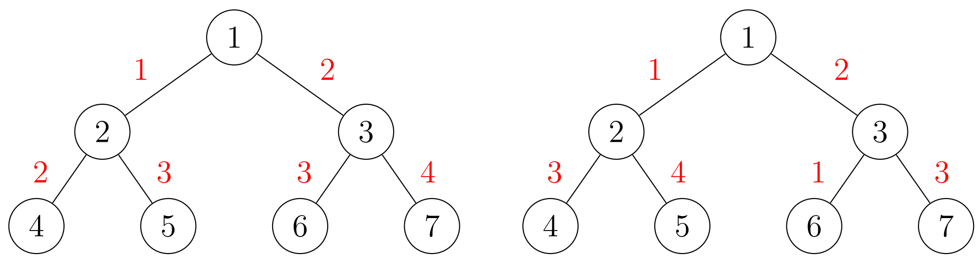

Figure 3.12: Isomorphic colorings

Thus there are two unique repetition-free edge-colorings of $T_{2,2}$ up to isomorphism.

□

3.2.3 Repetition-free 4-colorings of $T_{2,3}$

Lemma 3.2.4. *Every repetition-free 4-coloring of $T_{2,3}$ is isomorphic to one of the six colorings in Figures 3.13 - 3.18.*

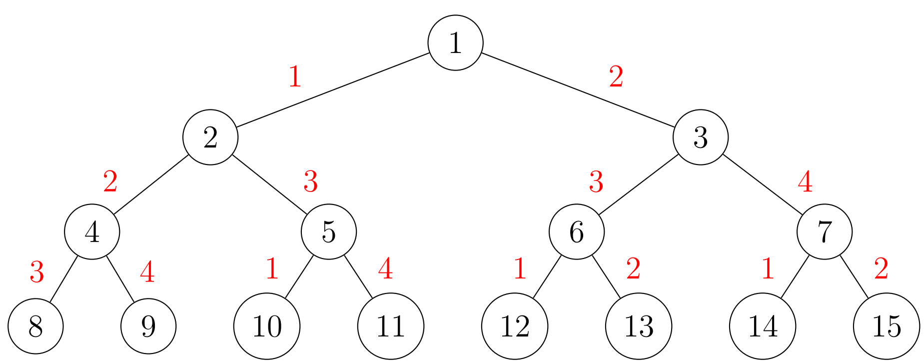

Figure 3.13

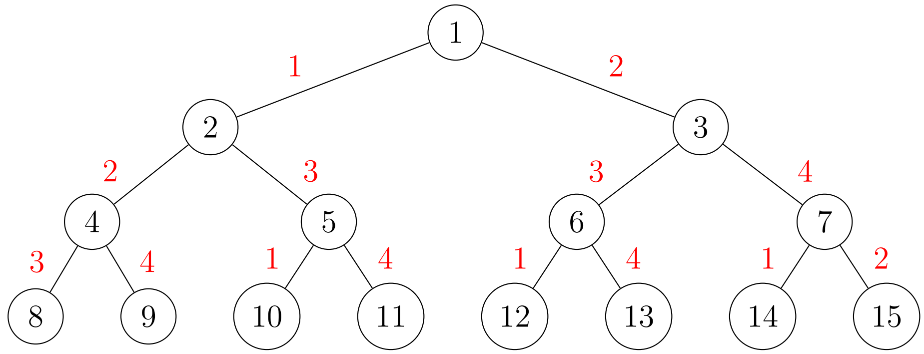

Figure 3.14

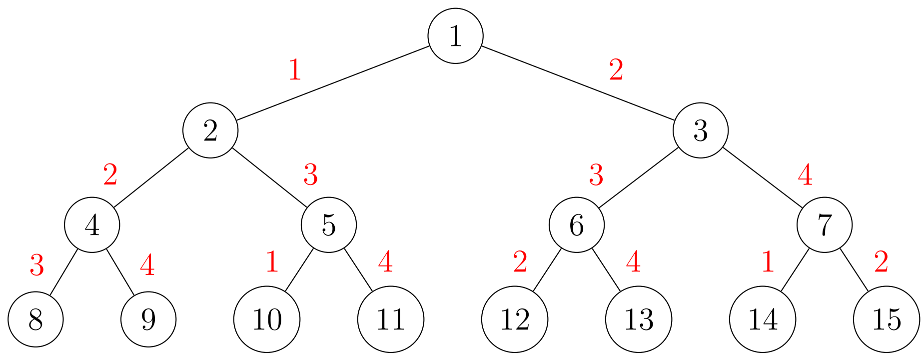

Figure 3.15

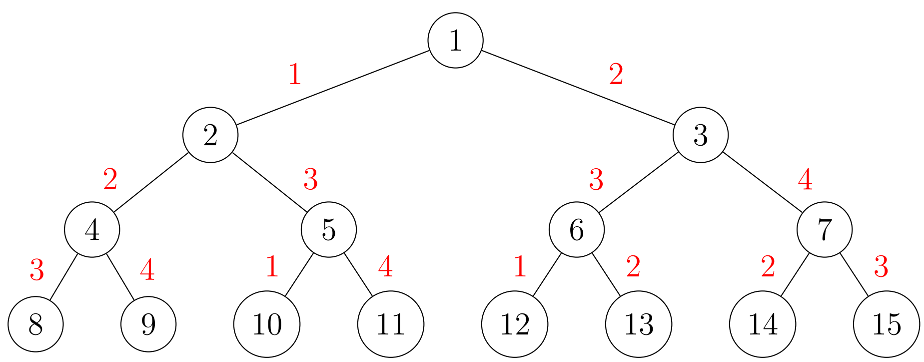

Figure 3.16

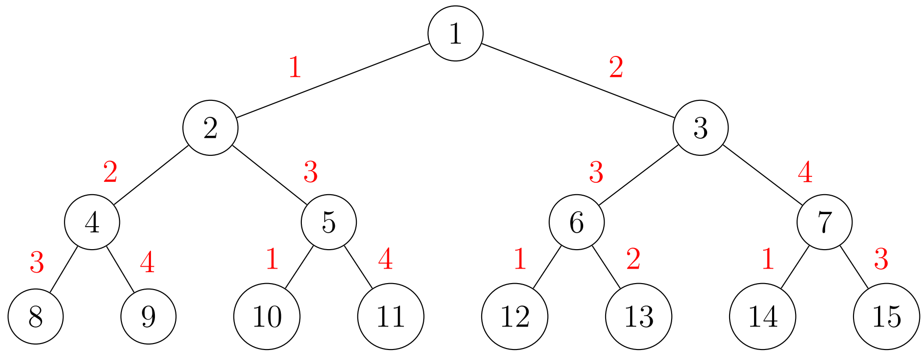

Figure 3.17

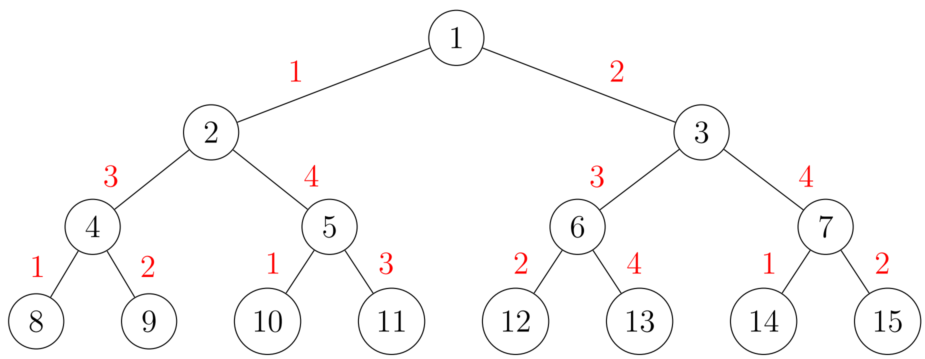

Figure 3.18

Proof. Like in the proof of Lemma 3.2.3 we will extend the colorings of $T_{2,2}$ given in Figures 3.9, 3.10 to $T_{2,3}$.

<u>Case 1</u>: Extension of $T_{2,2}$ from Figure 3.9. We can not use color 1 or color 2 on e_7, e_8 otherwise we have a repetition with color pattern 1212 or 22. So $c(e_7) = 3$ and $c(e_8) = 4$. If $c(e_9) = 2$ or $c(e_{10}) = 2$ then we have a repetition on 8,4,2,5,10 or 8,4,2,5,11 respectively that has color pattern 3232, so $c(e_9)$ and $c(e_{10})$ are forced. Without loss of generality $c(e_9) = 1$ and $c(e_{10}) = 4$.

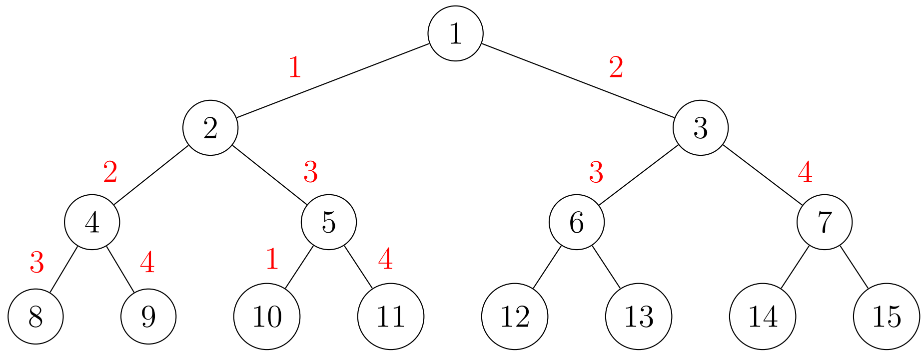

Figure 3.19

Let H' be a subgraph of Figure 3.19 such that $V(H') = \{3, 6, 7, 12, 13, 14, 15\}$ and $E(H') = \{e_5, e_6, e_{11}, e_{12}, e_{13}, e_{14}\}$. The repetition-free edge-coloring of H' must be isomorphic to either Figure 3.9 or Figure 3.10. Since color 3 and color 4 are no longer isomorphic we must try all isomorphic colorings of H', the only things we disregard is the ordering of the color on the leaves. That is the following color blocks for $c(e_{11}), c(e_{12}), c(e_{13}), c(e_{14})$ are all isomorphic $1212, 1221, 2121, 2112$. None of the resulting five nonisomorphic colorings cause repetitions so we have the following color blocks for the uncolored edges.

From Figure 3.10, $c(e_{11})c(e_{12})c(e_{13})c(e_{14}) : 1212$

From Figure 3.9, $c(e_{11})c(e_{12})c(e_{13})c(e_{14}) : 1412 \quad 2412 \quad 1223$ 1213

These are precisely the five color extensions of Figure 3.9, given in Figures 3.13-3.17.

<u>Case 2</u>: Extension of $T_{2,2}$ from Figure 3.10.

Observe that $T_{2,3}$ has $T_{2,2}$ as a subgraph H with $V(H) = \{2, 4, 5, 8, 9, 10, 11\}$ and $E(H) = \{e_3, e_4, e_7, e_8, e_9, e_{10}\}$ (see Figure 3.20). Since the edge-coloring of this subgraph needs to be repetition-free it must be isomorphic to Figure 3.9 or Figure 3.10.

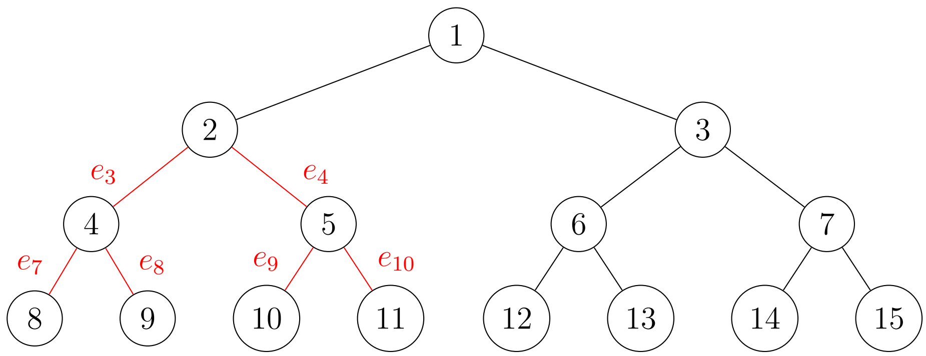

Figure 3.20: A drawing of $T_{2,3}$ with subgraph H in red

Suppose first that H has an edge-coloring isomorphic to Figure 3.9, where we may assume that the repeated color is 4, $f := \{(1,1),(2,2),(3,3),(4,5),(5,4),(6,6),(7,7)\}$ and the colors defined by: $1 \to 3$, $2 \to 4$, $3 \to 1$, and $4 \to 2$, which gives us Figure 3.21. Observe that $c(e_7) = 1$ or $c(e_7) = 2$ so we get non-isomorphic colorings depending on our choice of color 1 or color 2 but this turns out to be irrelevant for the rest of the proof.

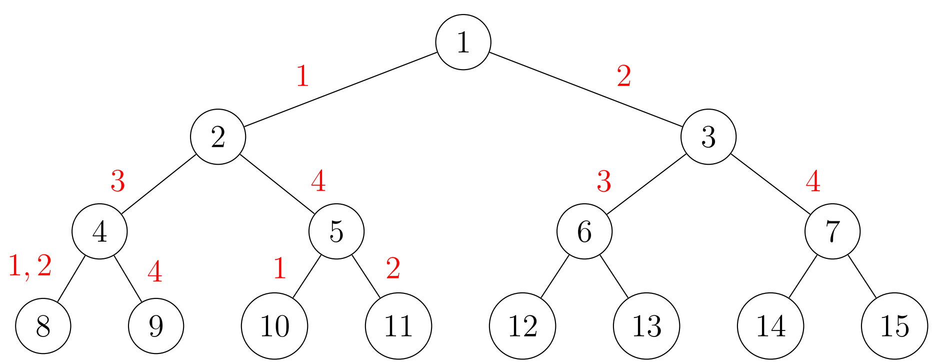

Figure 3.21

If $c(e_{13}) = 1$ or $c(e_{14}) = 1$ we have a repetition on $11, 5, 2, 1, 3, 7, 14$ or

$11, 5, 2, 1, 3, 7, 15$ respectively, so $c(e_{14})$ and $c(e_{15})$ are forced. Without loss of generality $c(e_{13}) = 2$ and $c(e_{14}) = 3$.

If $c(e_{11}) = 4$ or $c(e_{12}) = 4$ we have a repetition on $15, 7, 3, 6, 12$ or $15, 7, 3, 6, 13$ respectively, so $c(e_{11})$ and $c(e_{12})$ are forced. Without loss of generality $c(e_{11}) = 1$ and $c(e_{12}) = 2$.

We can now determine the color of e_7. If $c(e_7) = 2$ then we have a repetition on $8, 4, 2, 1, 3, 6, 12$ with color pattern 231231, so $c(e_7) = 1$.

This gives us Figure 3.22, which is isomorphic to the coloring given in Figure 3.18.

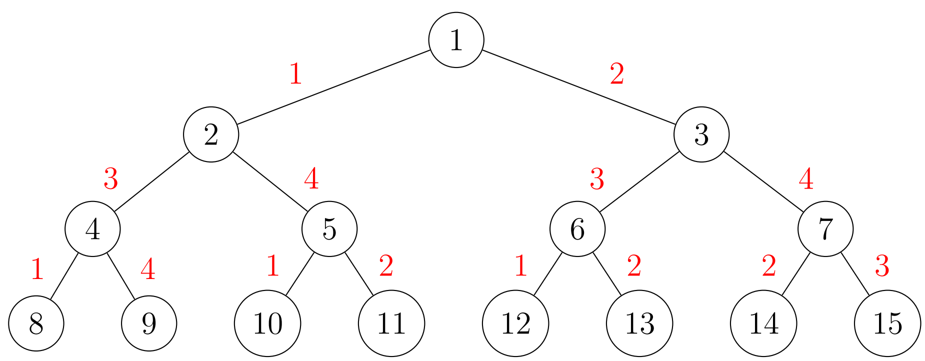

Figure 3.22

Now suppose that H has an edge-coloring isomorphic to Figure 3.10 with f as the identity and the colors defined by: $1 \to 3$, $2 \to 4$, $3 \to 1$, and $4 \to 2$, which gives us Figure 3.23.

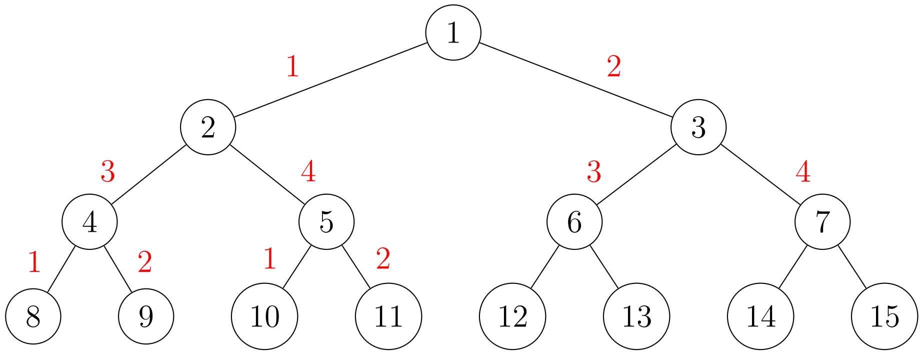

Figure 3.23

If $c(e_{13}) = 1$ or $c(e_{14}) = 1$ we have a repetition on $11, 5, 2, 1, 3, 7, 14$ or $11, 5, 2, 1, 3, 7, 15$ respectively that has color pattern 241241, so $c(e_{13})$ and $c(e_{14})$ are forced. Without loss of generality $c(e_{14}) = 2$ and $c(e_{15}) = 3$.

If $c(e_{11}) = 4$ or $c(e_{12}) = 4$ we have a repetition on $15, 7, 3, 6, 12$ or $15, 7, 3, 6, 13$ respectively that has color pattern 3434, so $c(e_{11})$ and $c(e_{12})$ are forced. Without loss of generality $c(e_{11}) = 1$ and $c(e_{12}) = 2$. This gives us Figure 3.24. However this causes a repetition on $12, 6, 3, 1, 2, 4, 9$ with color pattern 132132 so there is no repetition-free extension.

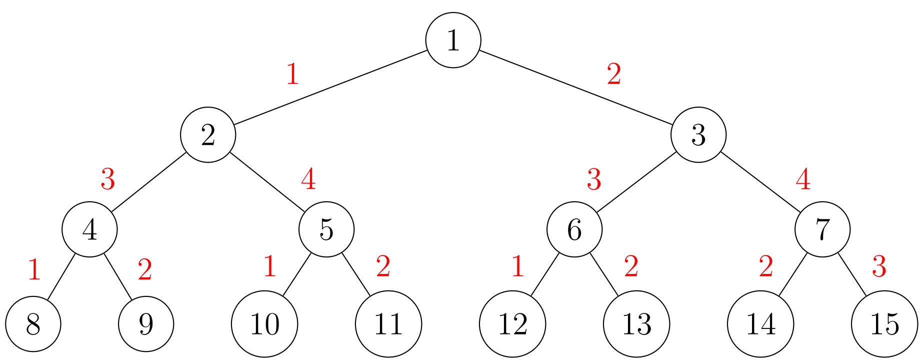

Figure 3.24

□

3.2.4 Repetition-free 4-colorings of $T_{2,4}$

Lemma 3.2.5. *There is no repetition-free 4-coloring of $T_{2,4}$*

Proof. We will show that none of the colorings in Figures 3.13-3.18 extends to $T_{2,4}$ in six cases. Forced colors on level four will appear in black. Before we begin cases 1-5 we make the following observations about an extension of Figure 3.19 and show that it is forced to Figure 3.25.

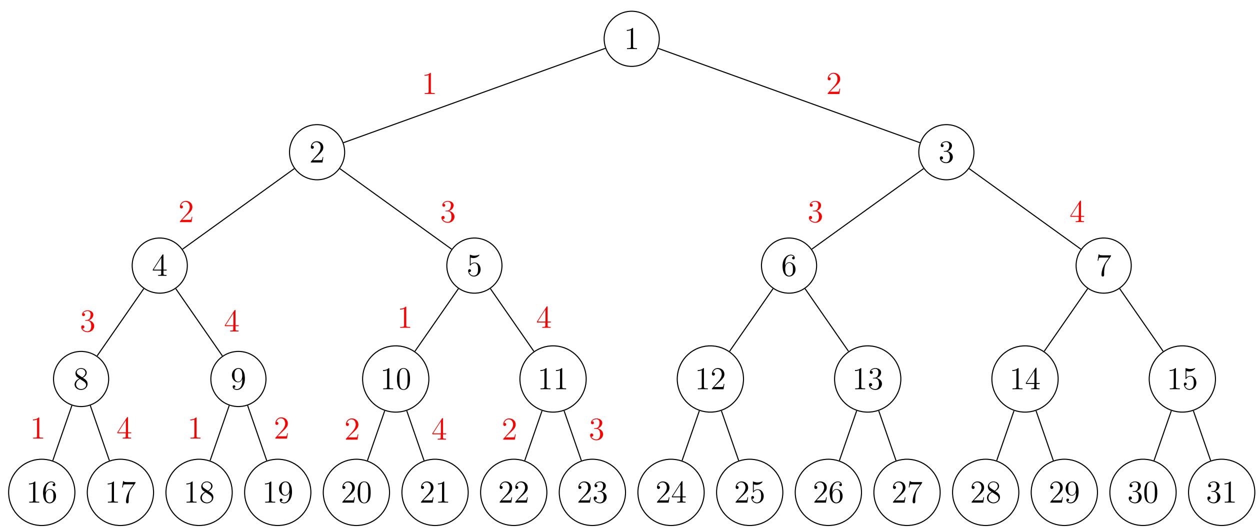

Figure 3.25: Forced colors of cases 1-5.

If $c(e_{15}) = 2$ or $c(e_{16}) = 2$ we have a repetition on $16, 8, 4, 2, 5$ or $17, 8, 4, 2, 5$ respectively that has color pattern 2323, so $c(e_{15})$ and $c(e_{16})$ are forced. Without loss of generality $c(e_{15}) = 1$ and $c(e_{16}) = 4$.

If $c(e_{17}) = 3$ or $c(e_{18}) = 3$ we have a repetition on $17, 8, 4, 9, 18$ or $17, 8, 4, 9, 19$ respectively that has color pattern 4343, so $c(e_{17})$ and $c(e_{18})$ are forced. Without loss of generality $c(e_{17}) = 1$ and $c(e_{18}) = 2$.

If $c(e_{19}) = 3$ or $c(e_{20}) = 3$ we have a repetition on $1, 2, 5, 10, 20$ or $1, 2, 5, 10, 21$ respectively that has color pattern 1313, so $c(e_{19})$ and $c(e_{20})$ are forced. Without loss of generality $c(e_{19}) = 2$ and $c(e_{20}) = 4$.

If $c(e_{21}) = 1$ or $c(e_{22}) = 1$ we have a repetition on $21, 10, 5, 11, 22$ or $21, 10, 5, 11, 23$ respectively that has color pattern 4141, so $c(e_{21})$ and $c(e_{22})$ are forced. Without loss of generality $c(e_{21}) = 2$ and $c(e_{22}) = 3$.

Case 1: Let us try to extend Figure 3.13 (see Figure 3.26 for the forced colors discussed next).

If $c(e_{29}) = 4$ or $c(e_{30}) = 4$ we have a repetition on $1, 3, 7, 15, 30$ or $1, 3, 7, 15, 31$ respectively that has color pattern 2424, so $c(e_{29})$ and $c(e_{30})$ are forced. Without loss of generality $c(e_{29}) = 1$ and $c(e_{30}) = 3$.

If $c(e_{25}) = 3$ or $c(e_{26}) = 3$ we have a repetition on $1, 3, 6, 13, 26$ or $1, 3, 6, 13, 27$ respectively that has color pattern 2323, so $c(e_{25})$ and $c(e_{26})$ are forced. Without loss of generality $c(e_{25}) = 1$ and $c(e_{26}) = 4$. However this causes a repetition on $27, 13, 6, 3, 7, 15, 31$ with color pattern 423423 so there is no repetition-free extension.

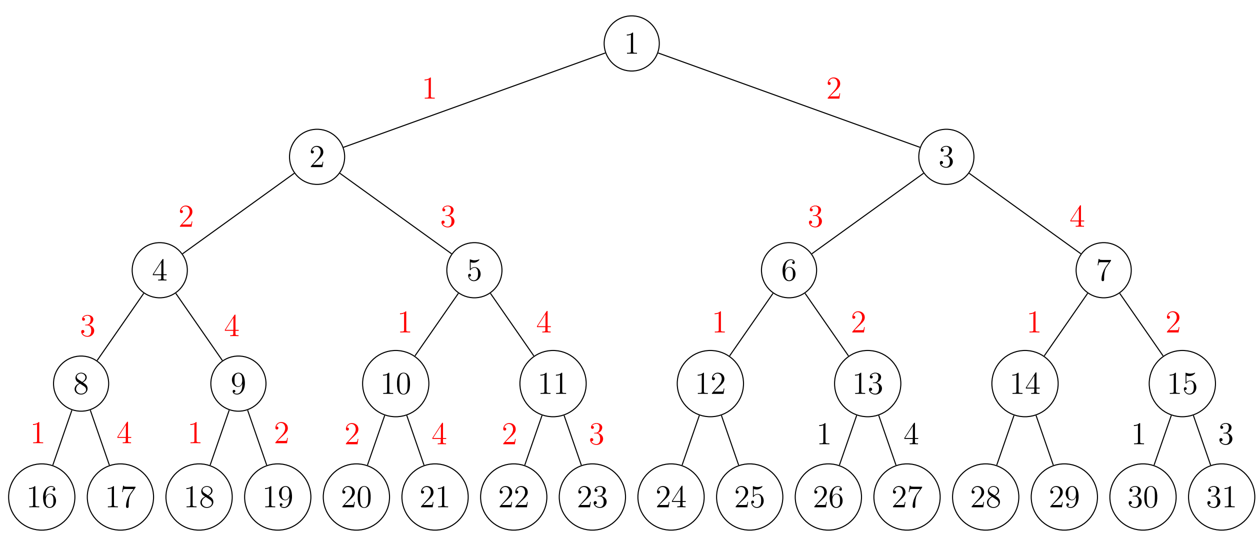

Figure 3.26: Colors forced in Case 1

<u>Case 2</u>: Let us try to extend Figure 3.14 (see Figure 3.27 for the forced colors discussed next).

If $c(e_{29}) = 4$ or $c(e_{30}) = 4$ we have a repetition on $1, 3, 7, 15, 30$ or $1, 3, 7, 15, 31$ respectively that has color pattern 2424, so $c(e_{29})$ and $c(e_{30})$ are forced. Without loss of generality $c(e_{29}) = 1$ and $c(e_{30}) = 3$. However this causes a repetition on $19, 9, 4, 2, 1, 3, 7, 15, 30$ with color pattern 24212421 so there is no repetition-free extension.

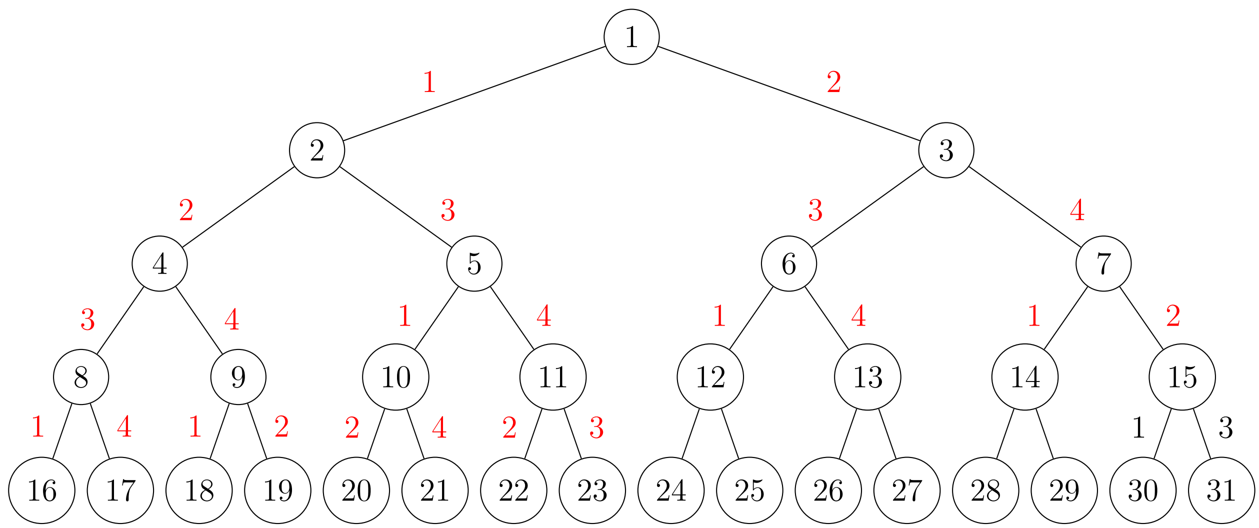

Figure 3.27: Colors forced in Case 2

<u>**Case 3**</u>: Let us try to extend Figure 3.15 (see Figure 3.2.4 for the forced colors discussed next).

If $c(e_{23}) = 3$ or $c(e_{24}) = 3$ we have a repetition on $1, 3, 6, 12, 24$ or $1, 3, 6, 12, 25$ respectively that has color pattern 2323, so $c(e_{23})$ and $c(e_{24})$ are forced. Without loss of generality $c(e_{23}) = 1$ and $c(e_{24}) = 4$.

If $c(e_{25}) = 2$ or $c(e_{26}) = 2$ we have a repetition on $25, 12, 6, 13, 26$ or $25, 12, 6, 13, 27$ respectively that has color pattern 2424, so $c(e_{25})$ and $c(e_{26})$ are forced. Without loss of generality $c(e_{25}) = 1$ and $c(e_{26}) = 3$. However this causes a repetition on $7, 3, 6, 13, 27$ with color pattern 4343 so there is no repetition-free extension.

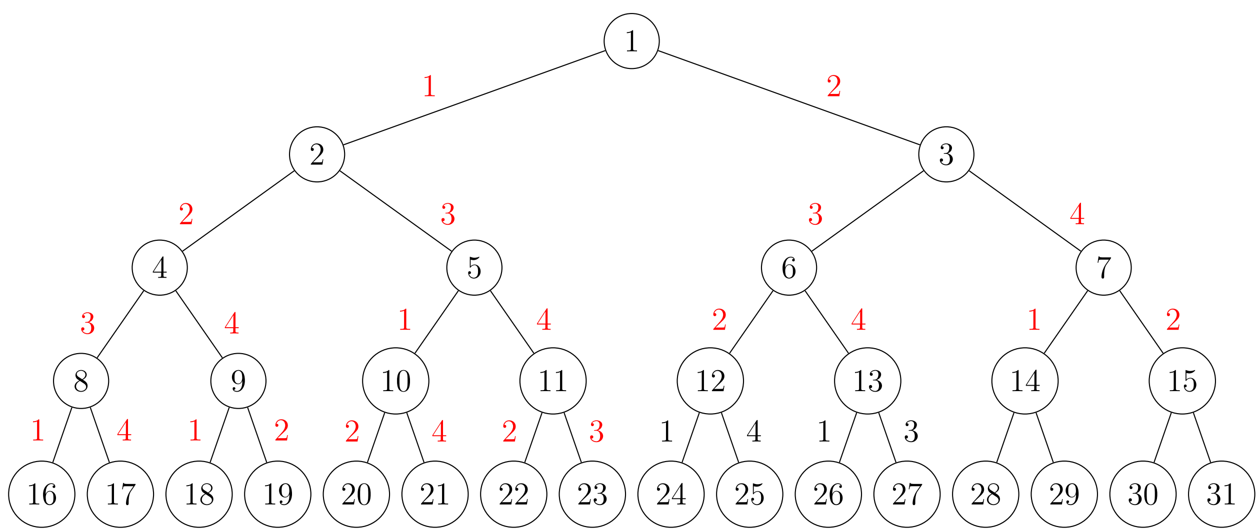

Figure 3.28: Colors forced in Case 3

Case 4: Let us try to extend Figure 3.16 (see Figure 3.29 for the forced colors discussed next).

If $c(e_{27}) = 4$ or $c(e_{28}) = 4$ we have a repetition on $1, 3, 7, 14, 28$ or $1, 4, 7, 14, 29$ respectively that has color pattern 2424, so $c(e_{27})$ and $c(e_{28})$ are forced. Without loss of generality $c(e_{27}) = 1$ and $c(e_{28}) = 3$.

If $c(e_{29}) = 2$ or $c(e_{30}) = 2$ we have a repetition on $29, 14, 7, 15, 30$ or $29, 14, 7, 15, 31$ respectively that has color pattern 3232, so $c(e_{29})$ and $c(e_{30})$ are forced. Without loss of generality $c(e_{29}) = 1$ and $c(e_{30}) = 4$. However this causes a repetition on $6, 3, 7, 15, 31$ with color pattern 3434 so there is no repetition-free extension.

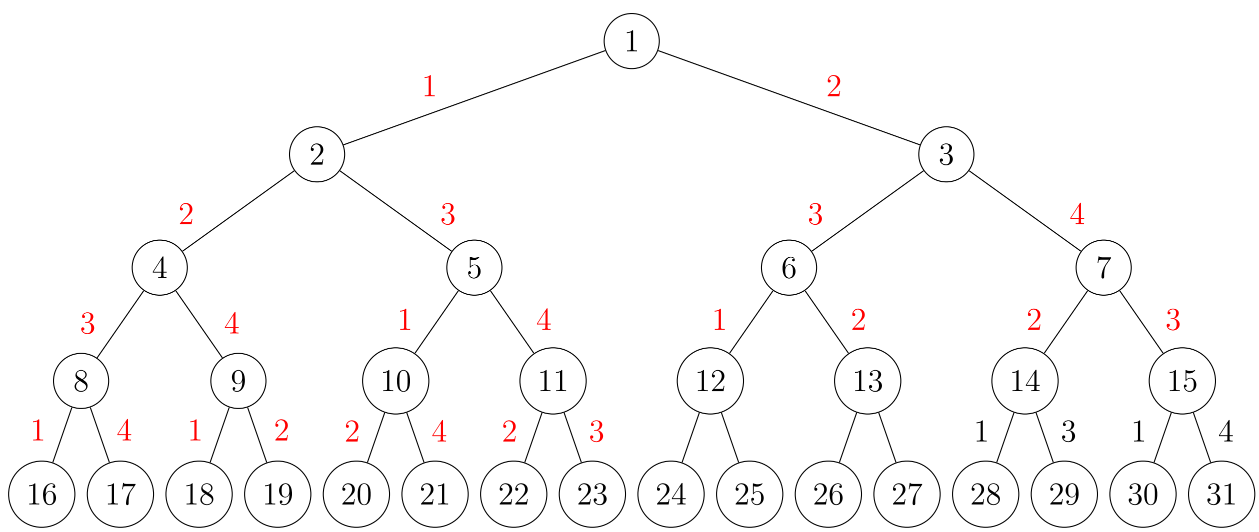

Figure 3.29: Colors forced in Case 4

<u>Case 5</u>: Let us try to extend Figure 3.17 (see Figure 3.30 for the forced colors discussed next).

If $c(e_{29}) = 1$ or $c(e_{30}) = 1$ we have a repetition on $22, 11, 5, 2, 1, 3, 7, 15, 30$ or $22, 11, 5, 2, 1, 3, 7, 15, 31$ respectively that has color pattern 24312431, so $c(e_{29})$ and $c(e_{30})$ are forced. Without loss of generality $c(e_{29}) = 2$ and $c(e_{30}) = 4$. However this causes a repetition on $6, 3, 7, 15, 31$ with color pattern 3434 so there is no repetition-free extension.

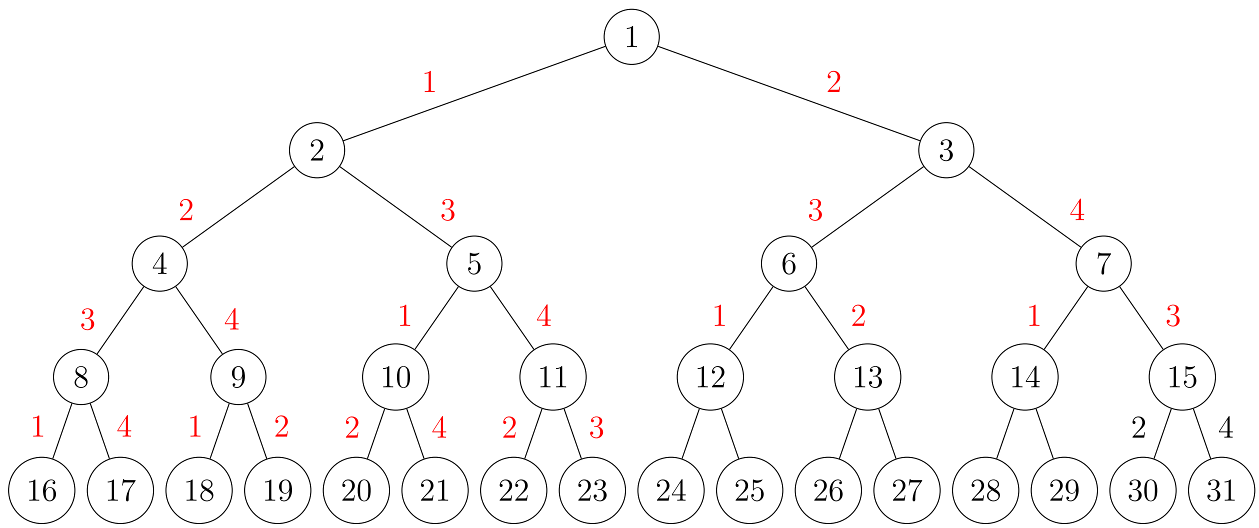

Figure 3.30: Colors forced in Case 5

Case 6: Let us try to extend Figure 3.18 (see Figure 3.31 for the forced colors discussed next).

If $c(e_{19}) = 4$ or $c(e_{20}) = 4$ we have a repetition on $1, 2, 5, 10, 20$ or $1, 2, 5, 10, 21$ respectively that has color pattern 1414, so $c(e_{19})$ and $c(e_{20})$ are forced. Without loss of generality $c(e_{19}) = 2$ and $c(e_{20}) = 3$.

If $c(e_{15}) = 3$ or $c(e_{16}) = 3$ we have a repetition on $1, 2, 4, 8, 16$ or $1, 2, 4, 8, 17$ respectively that has color pattern 1313, so $c(e_{15})$ and $c(e_{16})$ are forced. Without loss of generality $c(e_{15}) = 2$ and $c(e_{16}) = 4$. However this causes a repetition on $21, 10, 5, 2, 4, 8, 17$ with color pattern 314314 so there is no repetition-free extension.

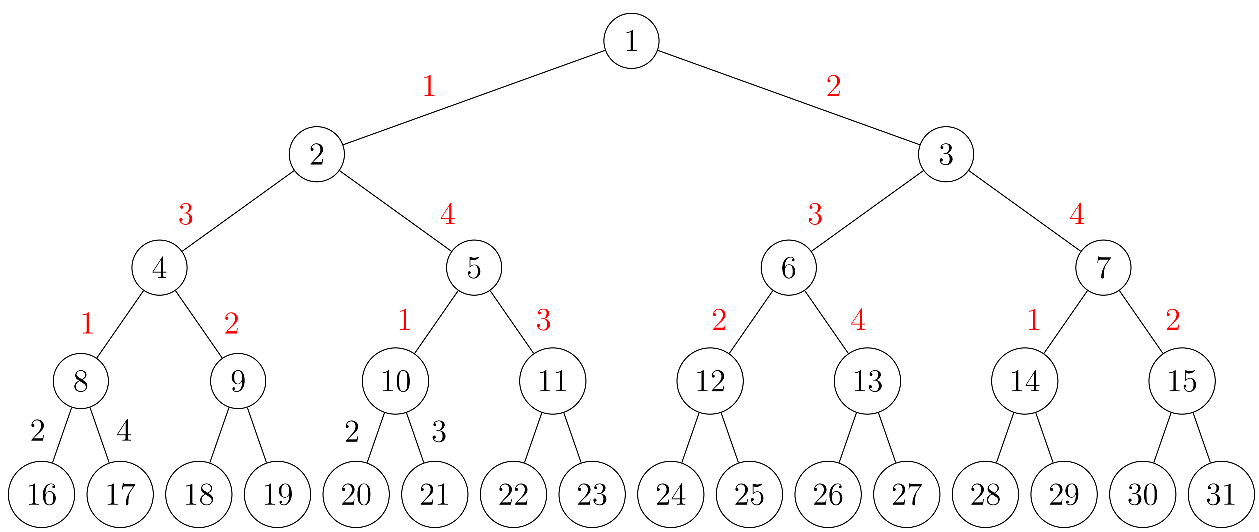

Figure 3.31: Colors forced in Case 6

□

3.2.5 Repetition-free 5-coloring of $T_{2,h}$

To finish the proof of Theorem 3.2.1 it suffices to give a repetition-free 5-coloring of $T_{2,10}$, since this will contain a 5-coloring for all $T_{2,h}$ with $3 < h < 11$. Using the computer programs described in the appendix we found the repetition-free 5-coloring of $T_{2,10}$ given in Example 3.2.6 . There are many repetition-free 5-colorings for $T_{2,4}$ and an exhaustive study of how these extend to $T_{2,h}$ for $h > 4$ does not seem viable.

Example 3.2.6. *Algorithm 3 in the appendix can be used to verify that the following color sequence of* $T_{2,10}$ *is repetition-free.*

*1, 2, 2, 3, 3, 4, 3, 4, 4, 5, 1, 2, 1, 2, 4, 1, 2, 1, 5, 2, 1, 2, 5, 4, 5, 1, 4, 2, 3,
5, 2, 1, 5, 3, 3, 5, 2, 3, 3, 1, 1, 4, 5, 4, 4, 3, 2, 1, 5, 1, 4, 3, 3, 5, 2, 3, 3,
5, 5, 2, 1, 2, 5, 3, 2, 3, 4, 1, 4, 2, 4, 2, 3, 4, 5, 3, 5, 1, 4, 2, 5, 2, 2, 4, 3,
5, 3, 2, 5, 3, 3, 1, 1, 2, 3, 5, 4, 2, 1, 2, 2, 3, 3, 5, 5, 2, 1, 5, 2, 4, 1, 5, 2,
4, 5, 1, 4, 1, 2, 1, 4, 1, 5, 2, 4, 3, 3, 2, 2, 1, 5, 4, 1, 5, 3, 5, 3, 2, 2, 1, 1,
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5, 5, 1, 5, 3, 4, 2, 5, 1, 1, 4, 4, 2, 4, 3, 3, 5, 4, 3, 1, 4, 3, 5, 3, 5, 3, 5, 3,*

*4, 1, 5, 3, 4, 3, 5, 3, 4, 1, 5, 5, 3, 4, 2, 5, 4, 1, 5, 5, 1, 1, 3, 1, 4, 5, 2, 1,
3, 5, 4, 3, 5, 2, 1, 2, 4, 4, 3, 1, 5, 3, 2, 2, 5, 4, 3, 3, 1, 1, 2, 1, 5, 3, 2, 1,
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Chapter 4

Upper bounds

To study the bounds and coloring concepts we introduce in this chapter it is helpful to be able to go between finite k-ary trees $T_{k,h}$ and their infinite counterpart T_k.

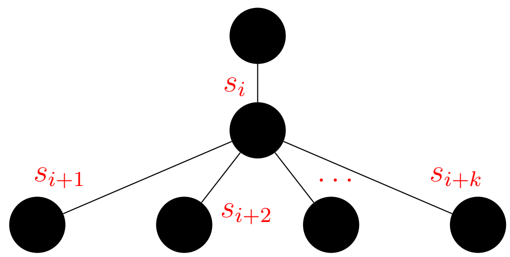
Definition 4.0.1. *The* ***infinite*** k***-ary tree*** T_k *is the tree with vertex set* $\mathbb{N} = \{1, 2, 3, \dots\}$ *where 1 is the root vertex and every vertex* i *has children* $(i-1)k+2, (i-1)k+3, \dots, (i-1)k+k+1$.

The labels of the children are just like in Section 2.3.1, and many other concepts carry over. Every non-root vertex i has parent $p(i) = \lfloor (i+k-2)/k \rfloor$ and k children, so that apart from 1 every vertex has degree $k+1$. We will go between T_k and $T_{k,h}$ depending on which is the most convenient.

4.1 Derived colorings

In this section we will describe a useful strategy for obtaining edge-colorings of $T = T_{k,h}$ and $T = T_k$ from a sequence S of length at least kh or an infinite sequence S, respectively. So let $S = s_1, s_2, \dots$ be given. The edge-coloring of

T **derived** from S is obtained as follows: Label T as previously explained. The children of the root get colors $s_1, s_2, \ldots, s_k$ with s_1 assigned to the left edge, continue in this fashion until s_k is on the right edge. Let $v \in V(T)$. If the edge between v and the parent of v in T has color s_i then the edges between v and its children receive colors $s_{i+1}, s_{i+2}, \ldots, s_{i+k}$ as shown below, with s_{i+1} on the left edge and s_{i+k} on the right edge. The length of S must be at least kh as each level will use k entries of S more than the previous level.



Example 4.1.1. *The edge-coloring of* $T_{2,3}$ *derived from* $S = 1, 2, 3, 4, 1, 2$ *is shown in Figure 4.1 which is isomorphic to the edge-coloring in Figure 3.14 with isomorphism,*

$$f := \{(10, 11), (11, 10), (12, 13), (13, 12)\} \cup \{(i, i) : i = 1, 2, \ldots, 9, 14, 15\}.$$

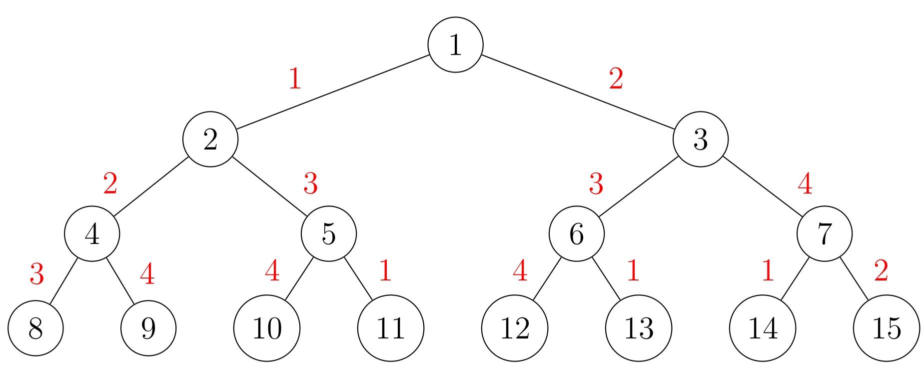


Figure 4.1

4.2 k-special sequences

The aim of this section is to characterize sequences whose derived coloring produces repetition-free edge-colorings on T_k.

Definition 4.2.1. *A sequence $S = s_1, s_2, \dots$ is called k-**special** if there is no sequence of indices $i_1, i_2, \dots, i_{2r}$ and $1 < m \leq 2r$ such that the following four conditions hold:*

a) $s_{i_1}, s_{i_2}, \dots, s_{i_{2r}}$ is a repetition

b) $i_1 > i_2 > \cdots > i_m < i_{m+1} < i_{m+2} < \cdots < i_{2r}$

c) $|i_j - i_{j+1}| \leq k$ for all j with $1 \leq j < 2r$

d) $i_{m+1} < i_m + k$ if $m < 2r$

Theorem 4.2.2. *A sequence is 1-special if and only if it is repetition-free.*

Proof. Let $S = s_1, s_2, \dots$ be a sequence. In both cases we will proceed by contrapositive.

($\Rightarrow$) Suppose that S contains a repetition $s = s_k, s_{k+1}, \dots, s_{k+2r-1}$ as a block. Let $i_j = k+2r-j$ then $i_1 > i_2 > \cdots > i_{2r}$ and $|i_j - i_{j+1}| = 1$. Hence S contains a repetition such that the four conditions hold with $m = 2r$. Thus S is not 1-special.

($\Leftarrow$) Suppose that S is not 1-special. S contains a repetition $s = s_{i_1}, s_{i_2}, \dots, s_{i_{2r}}$ such that the four conditions hold for $k = 1$. Since $|i_j - i_{j+1}| \leq 1$ it means the terms of s are consecutive. Note that $m = 2r$ as $i_m < i_{m+1}$ so $i_1 > i_2 > \cdots > i_{2r}$. Thus S contains a repetition as a block, so S is not repetition-free.

□

Theorem 4.2.3. *If a sequence is k-special then it is $(k-1)$-special.*

Proof. We proceed by contrapositive. Let S be a sequence that is not $(k-1)$-special. By definition S contains a repetition $s = s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ such that the four conditions hold. Observe that $|i_j - i_{j+1}| \leq k - 1 < k$. If $m < 2r$ then $i_{m+1} < i_m + k - 1 < i_m + k$. Thus S is not k-special as desired. □

Theorem 4.2.4. *An infinite sequence S is k-special if and only if the edge-coloring of T_k derived from S is repetition-free.*

Proof. (⇒) Suppose that a k-special sequence S creates a repetition on a path $P = v_0, v_1, \ldots, v_{2r}$ in T_k, that is $R = c(v_0v_1), c(v_1v_2), \ldots, c(v_{2r-1}v_{2r})$ satisfies $c(v_iv_{i+1}) = c(v_{i+r}c_{i+r+1})$ for $0 \leq i \leq r-1$. Observe that $c(v_jv_{j+1}) = s_{i_{j+1}}$ where $0 \leq j \leq 2r-1$, for some $s_{i_{j+1}} \in S$. There are two possibilities; $v_0, v_1, \ldots, v_{2r}$ is monotone or it has a single turning point.

Case 1: Suppose $v_0, v_1, \ldots, v_{2r}$ is monotone.

If $v_0, v_1, v_2 \ldots, v_{2r}$ is monotone then we may assume $v_0 > v_1 > v_2 > \cdots > v_{2r}$. Since $v_j > v_{j+1}$ we know that v_j is the child of v_{j+1} so we have that $i_j > i_{j+1}$ and $|i_j - i_{j+1}| \leq k$. The subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ is a repetition, which contradicts the fact that S is k-special, where we let $m = 2r$.

Case 2: Suppose $v_0, v_1, \ldots, v_{2r}$ has a turning point for some m with $0 < m < 2r$. By definition $v_0 > v_1 > \cdots > v_{m-1} > v_m < v_{m+1} < \cdots < v_{2r}$ where v_m is the turning point and without loss of generality $v_{m-1} < v_{m+1}$. Observe that $v_0, v_1, \ldots, v_m$ is going up and $v_m, v_{m+1}, \ldots, v_{2r}$ is going down. Let $c(v_jv_{j+1}) = s_{i_{j+1}}$. Thus $i_1 > i_2 > \cdots > i_{m-1} > i_m < i_{m+1} < \cdots < i_{2r}$. Since $v_{j-1} > v_j > v_{j+1}$ for $1 \leq j < m$ we know that v_j is the child of v_{j+1} and the parent of v_{j-1} so we have $i_j > i_{j+1}$ and $|i_j - i_{j+1}| \leq k$. Similarly, since $v_{j-1} < v_j < v_{j+1}$ for $m < j < 2r$ we know that v_j is the child of v_{j-1} and the parent of v_{j+1} so $i_j < i_{j+1}$ and $|i_j - i_{j+1}| \leq k$. Finally, since v_m is the turning point it is the parent of v_{m-1} and v_{m+1} so $|i_m - i_{m+1}| < k$ and $i_m < i_{m+1}$ since we assumed $v_{m-1} < v_{m+1}$. The subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ is a repetition, which contradicts the fact that S is k-special.

($\Leftarrow$) We proceed by contrapositive. Let S be a sequence that is not k-special. By definition S contains a repetition $s = s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ such that the four conditions hold. We will show that there is a path on vertices $v_0, v_1, v_2, \ldots, v_{2r}$ with $c(v_j v_{j+1}) = s_{i_{j+1}}$ because then the color pattern $c(v_0v_1), c(v_1v_2)\ldots, c(v_{2r-1}v_{2r})$ is a repetition in the derived edge-coloring of T_k. Observe that if v' is the left child (see Definition 3.1.9) of a vertex v and $c(vp(v)) = s_\alpha$, then $c(vv') = s_{\alpha+1}$. From this point onward v' will be the left child of v.

If $m = 2r$ then we start at the root and successively go to the left child of the current vertex until we find a vertex v_{2r} such that $c(v_{2r}v'_{2r}) = s_{i_{2r}}$ and let $v_{2r-1} = v'_{2r}$. Let v_{2r-2} be the child of v_{2r-1} with $c(v_{2r-1}v_{2r-2}) = s_{i_{2r-1}}$ (this exists since $|i_j - i_{j+1}| \leq k$). We continue in this way until we have found v_0. Now observe that the color pattern of $v_0, v_1, \ldots, v_{2r}$ is $s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ as desired.

If $m < 2r$ then we start at the root and successively go to the left child of the current vertex until we find a vertex v_m such that $c(v_m v'_m) = s_{i_m}$ and let $v_{m-1} = v_{m'}$. Let v_{m+1} be the child of v_m with $c(v_m v_{m+1}) = s_{i_{m+1}}$ (this exists since $i_m < i_{m+1} < i_m + k$). Now, for $0 \leq p \leq (m-1)$ we successively find a child v_{p-1} of v_p such that $c(v_p v_{p-1}) = s_{i_p}$. The existence of v_{p-1} is guaranteed by the fact $|i_p - i_{p-q}| \leq k$ as in the case $m = 2r$. For $m+1 \leq q \leq 2r$ we successively find a child v_{q+1} of v_q such that $c(v_q v_{q+1}) = s_{i_{q-1}}$ which we can do since $|i_q - i_{q+1}| \leq k$. Now observe that the color pattern of $v_0, v_1, \ldots, v_{2r}$

is $s_{i_1}, s_{i_2}, \ldots, s_{i_{2r}}$ as desired. □

4.3 Finitely defined properties of sequences

Theorem 4.2.4 characterizes the sequences whose derived colorings of T_k are repetition-free. In this section we investigate the situation for $T_{k,h}$.

Lemma 4.3.1. *If a sequence $S = s_1, s_2, \ldots, s_n$ on symbols $\{1, 2, \ldots, t\}$ is k-special, then the infinite sequence $S' = s_1, s_2, \ldots, s_n, t+1, t+2, \ldots$ is k-special.*

Proof. We proceed by contrapositive. Suppose S' is not k-special. By definition there is a sequence of indices $i_1, i_2, \ldots, i_{2r}$ that satisfies the four conditions in Definition 4.2.1 for S'. If $i_j \leq n$ for all j, then this yields the desired sequence of indices showing that S is not k-special. So we may suppose that $i_j > n$ for some j. Since these indices yield a repetition in S, and s_{i_j} is unique in S it follows that we can assume that $j \leq r$ and that $i_{j+r} = i_j$. It follows that $j < m < j + r$. If $m \leq r$ it follows that $i_{m+r} > i_m$ and since $s_{i_m} = s_{i_{m+r}}$ we conclude that $i_m, i_{m+r} \leq n$. This contradicts the fact that $n < i_{j+r} < i_{m+r}$. In the case when $m > r$ we get the similar contradiction that $n < i_j < i_{m-r}$.

□

Theorem 4.3.2. *If a sequence $S = s_1, s_2, \ldots, s_{kh}$ on t symbols is k-special then $\pi'(T_{k,h}) \leq t$.*

Proof. We may assume without loss of generality that the symbols being used in S are $\{1, 2, \ldots, t\}$. Consider the infinite sequence $S' = s_1, s_2, \ldots, s_{kh}, t+1,$

$t+2, \ldots$ which is k-special by Lemma 4.3.1. By Theorem 4.2.4 the edge-coloring of T_k derived from S' is repetition-free. Since $T_{k,h}$ is a subgraph of T_k it has a repetition-free edge-coloring that is derived from $s_1, s_2, \ldots, s_{kh} = S$ so the result follows. □

Theorem 4.3.2 is the finite version of Theorem 4.2.4, except that in Theorem 4.2.4 we have equivalence, whereas Theorem 4.3.2 is only an implication. The corresponding converse statement that if a derived coloring is repetition-free, then the sequence must have been k-special does not need to hold, as the next example illustrates.

Example 4.3.3. *Example 4.1.1 shows that the coloring of* $T_{2,3}$ *derived from* $S = 1,2,3,4,1,2$ *is repetition-free. However, the sequence* S *is not 2-special, because the index-sequence* $3,1,2,3,5,6$ *satisfies the four conditions from Definition 4.2.1 and yields the repetition* $3,1,2,3,1,2$. *The coloring of* T_2 *derived from the extended sequence* $1,2,3,4,1,2,5,6,7,8,\ldots$ *as in Lemma 4.3.1 would contain a path of length six with color pattern* $3,1,2,3,1,2$.

Observe that if there is a repetition-free m-edge-coloring of T_k, then there is obviously a repetition-free m-edge-coloring of $T_{k,h}$ for arbitrary h. To conclude this section we will prove that the converse is also true. Hence studying T_k and the corresponding infinite sequences is equivalent to studying general upper bounds on $\pi'(T_{k,h})$.

Recall that a **block** of a sequence is a subsequence of consecutive terms. An **initial block** of a sequence is a block of finite length whose first entry is the first entry from the sequence. In what follows we will assume that all sequences consist of symbols from a fixed finite alphabet Σ.

Definition 4.3.4. *A property P of a sequence over Σ is called* ***finitely defined*** *if there is a (possibly infinite) list $\mathcal{F}$ of finite sequences such that a sequence has property P if and only if it does not contain an element of $\mathcal{F}$ as an initial block.*

Example 4.3.5. *Since repetitions are finite we can see that the property of being repetition-free is finitely defined. Another example of such a property is being k-special.*

Theorem 4.3.6 will allows us to work with sequences of arbitrary length or infinite length as we desire.

Theorem 4.3.6. *Let P be a finitely defined property. There is an infinitely long sequence with property P over a given finite alphabet if and only if there are arbitrarily long sequences with property P.*

Proof. ($\Rightarrow$) Suppose that there is an infinitely long sequence with property P. Observe that any initial block of a infinite sequence with property P will also have property P as it will not contain any element of $\mathcal{F}$. We obtain arbitrarily long sequences with property P by taking initial blocks of desired length from the infinitely long sequence.

($\Leftarrow$) Suppose that there are arbitrarily long sequences $S_1, S_2, \ldots$ with property P. We will recursively define infinite sets of indices $I_0 \supseteq I_1 \supseteq I_2, \ldots$ with the property that all sequences S_i with $i \in I_k$ have the same entries in the first k positions. We start by letting $I_0 = \mathbb{N}$ trivially as we are not requiring any of the positions to have the same entries. Now suppose that I_j is an infinite set with the desired properties. Since our alphabet is finite but I_j is infinite, some value σ must repeat infinitely often in position $j+1$ of S_i

with $i \in I_j$. Now $I_{j+1} = \{i \in I_j : (j+1)\text{st entry of } S_i \text{ is } \sigma\}$ has the desired properties.

Define an infinite sequence $S = s_1, s_2, \ldots$ by letting s_j be the value that all sequences S_i with $i \in I_j$ have in their j^{th} position. Suppose for the sake of contradiction that S does not have property P. Consider an initial block $B = s_1, s_2, \ldots, s_m$ from S that is in $\mathcal{F}$, and let S_i be any sequence with $i \in I_m$. By definition of I_m, S_i must contain B, and thus S_i does not have property P, a contradiction. □

Theorem 4.3.6 yields the following result.

Corollary 4.3.7. *Let k, t be positive integers. $\pi'(T_k) \leq t$ if and only if $\pi'(T_{k,h}) \leq t$ for infinitely many h.*

Proof. ($\Rightarrow$) Since $T_{k,h}$ is a subgraph of T_k it immediately follows that $\pi'(T_{k,h}) \leq \pi'(T_k) \leq t$ for all h.

($\Leftarrow$) We begin by identifying a t-edge coloring of $T_{k,h}$ and T_k with a color sequence S on $\{1, 2, \ldots, t\}$ as in Definition 3.1.1. Let P be the property that S is the color sequence of a repetition-free edge-coloring of a perfect k-ary tree. We can see that property P of S is finitely defined. Any path in T that has a color pattern that is a repetition would contain a vertex v_m of highest index. We may avoid this particular repetition by including the corresponding initial blocks $c(e_1), c(e_2), \ldots, c(e_{m-1})$ in $\mathcal{F}$.

Now $\pi'(T_{k,h}) \leq t$ means that we can find a color sequence of length at least h^k derived from a repetition-free coloring of $T_{k,h}$ with colors $\{1, 2, \ldots, t\}$, that has property P. By Theorem 4.3.6 there is an infinite sequence over $\{1, 2, \ldots, t\}$ that has property P, and thus $\pi'(T_k) \leq t$ as desired. □

4.4 k-fold extensions

Definition 4.4.1. *If $A = a_1, a_2, \ldots$ is a sequence and $k > 1$ then the k**-fold extension** of A is $A^{(k)} := a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(k)}, a_2^{(1)}, a_2^{(2)}, \ldots$.*

Example 4.4.2. *If $A = 1, 2, 4$, then its 3-fold extension is*
$A^{(3)} = 1^{(1)}, 1^{(2)}, 1^{(3)}, 2^{(1)}, 2^{(2)}, 2^{(3)}, 4^{(1)}, 4^{(2)}, 4^{(3)}$.

Definition 4.4.3. *If $a_{i_1}^{(j_1)}, a_{i_2}^{(j_2)}, a_{i_3}^{(j_3)}, \ldots, a_{i_{2r}}^{(j_{2r})}$ is a sequence of elements from $A^{(k)}$ then the **reduced sequence** is obtained by erasing the superscripts. Merging identical consecutive terms of the reduced sequence yields the **underlying sequence.***

Example 4.4.4. *The sequence $a_1^{(2)}, a_2^{(1)}, a_2^{(3)}, a_2^{(5)}, a_3^{(1)}, a_3^{(3)}$ has reduced sequence $a_1, a_2, a_2, a_2, a_3, a_3$ and underlying sequence a_1, a_2, a_3.*

Lemma 4.4.5. *If A is repetition-free and $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ in $A^{(k)}$ then $a_p = a_{p'}$ implies either $p = p'$ and $q < q'$ or $p' \geq p + 2$. In the latter case $a_p^{(q)}$ and $a_{p'}^{(q')}$ are distance at least $k + 1$ in $A^{(k)}$.*

Proof. Suppose that $a_p = a_{p'}$. Since $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ in $A^{(k)}$ it follows that $p' = p$ or $p' > p$.

Case 1: Suppose $p' = p$. We created $A^{(k)}$ by replacing a_p in A with $a_p^{(1)}, a_p^{(2)}, a_p^{(3)}, \ldots, a_p^{(k)}$ so if $a_p^{(q)}$ precedes $a_{p'}^{(q')} = a_p^{(q')}$ in $A^{(k)}$ it follows that $q < q'$.

Case 2: Suppose $p' > p$. Since A is repetition-free it follows that $p' > p + 1$, as otherwise a_p, a_{p+1} would be a repetition. If $p' = p + 2$ then the closest $a_p^{(q)}$ and $a_{p'}^{(q')}$ can be is $q = k$ and $q' = 1$ with the terms $a_{p+1}^{(1)}, a_{p+1}^{(2)}, a_{p+1}^{(3)}, \ldots, a_{p+1}^{(k)}$ between them. Thus $a_p^{(q)}$ and $a_{p'}^{(q')}$ are distance at least $k + 1$ in $A^{(k)}$. □

Lemma 4.4.6. *If $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ in $A^{(k)}$ by at most k and $a_p \neq a_{p'}$, then $1 \leq q' \leq q$. Moreover $q' = q$ if and only if $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ by exactly k.*

Proof. If $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ by at most k but $a_p \neq a_{p'}$, then $p' = p + 1$. Thus there are $(k - q) + q' - 1$ terms between $a_p^{(q)}$ and $a_{p'}^{(q')}$ in $A^{(k)}$. Since $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ in $A^{(k)}$ by at most k, $k - q + q' - 1 \leq k - 1$ implies $q' \leq q$. Observe that $a_p^{(q)}$ precedes $a_{p'}^{(q')}$ by exactly k if and only if $k-q+q'-1 = k-1$ which implies that $q = q'$. □

These Lemmas allow us to prove the following general result.

Theorem 4.4.7. *If A is a repetition-free and palindrome-free sequence then $A^{(k)}$ is k-special.*

Proof. Suppose aiming for a contradiction that for $S = A^{(k)}$ there is a sequence sequence of indices $i_1, i_2, \ldots, i_{2r}$ that satisfies the four conditions from Definition 4.2.1 for S. Thus $S' = s_{i_1}, s_{i_2}, s_{i_3}, \ldots, s_{i_{2r}}$ is a repetition where $s_{i_j} = a_{p_j}^{(q_j)}$.

Case 1: Suppose that $m = 2r$. Observe that since S' is a repetition its reduced sequence S'' is also a repetition. Moreover, since $|i_j - i_{j+1}| \leq k$ it follows from Lemma 4.4.5 that consecutive terms in S'' have the same underlying symbol $a_{p_j} = a_{p_{j+1}}$ only when $p_j = p_{j+1}$. Therefore if we merge terms in the reduced sequence that are identical and consecutive we see that the underlying sequence of S' is a repetition in A which is a contradiction.

Case 2: Suppose that $m > r$. If $a_{p_m} = a_{p_{m+1}}$ then $p_m = p_{m+1}$. By Lemma 4.4.5, $q_m < q_{m+1}$. However since S' is a repetition $a_{p_{m-r}} = a_{p_{m-r+1}}$ and $q_{m-r} < q_{m-r+1}$ which is a contradiction to Lemma 4.4.5 and the fact that $i_{m-r} > i_{m-r+1}$. So it must be that $a_{p_m} \neq a_{p_{m+1}}$. Observe that $|i_m - i_{m+1}| < k$

so by Lemma 4.4.6 $q_{m+1} < q_m$. However since S' is a repetition $a_{p_{m-r}} \neq a_{p_{m-r+1}}$ and $q_{m-r} < q_{m-r+1}$ which is a contradiction to Lemma 4.4.6 and the fact that $i_{m-r} > i_{m-r+1}$.

Case 3: Suppose that $m = r$. Since S' is a repetition for $1 \leq t \leq r$ we have $s_{i_t} = s_{i_{r+t}}$ so $a_{p_j}^{(q_j)} = a_{p_{j+r}}^{(q_{j+r})}$ implies $q_j = q_{j+r}$. For $1 < j \leq m$ if $a_{p_j} = a_{p_{j-1}}$ then by Lemma 4.4.5 we have $p_j = p_{j-1}$ and $q_j < q_{j-1}$. However since S' is a repetition $a_{p_{j+r}} = a_{p_{j+r-1}}$ and $q_{j+r} < q_{j+r-1}$ which is a contradiction to Lemma 4.4.5. So it must be that $a_{p_j} \neq a_{p_{j-1}}$. It then follows from Lemma 4.4.6 that $q_{j-1} \leq q_j$. Observe that since S' is a repetition $a_{p_{j+r}} \neq a_{p_{j+r-1}}$ so by Lemma 4.4.6 $q_{j+r-1} \leq q_{j+r}$. Thus $q_{j-1} = q_j$, so $q_m = q_{m-1} = \cdots = q_1 = q_{m+1}$. Since $q_m = q_{m+1}$ we must either have $a_{p_m} = a_{p_{m+1}}$ (contradicting Lemma 4.4.5) or $a_{p_m} \neq a_{p_{m+1}}$ (contradicting Lemma 4.4.6 and $i_{m+1} < i_m + k$).

Case 4: Suppose that $m < r$. For $1 < j \leq m$ if $a_{p_j} = a_{p_{j-1}}$ then $p_j = p_{j-1}$ so by Lemma 4.4.5 $q_j < q_{j-1}$. However since S' is a repetition $a_{p_{j+r}} = a_{p_{j+r-1}}$ and $q_{j+r} < q_{j+r-1}$ which is a contradiction to Lemma 4.4.5 and $i_{j+r-1} < i_{j+r}$. So it must be that $a_{p_j} \neq a_{p_{j-1}}$. It then follows from Lemma 4.4.6 that $q_{j-1} \leq q_j$ since $i_{j-1} > i_j$. Observe that since S' is a repetition $a_{p_{j+r}} \neq a_{p_{j+r-1}}$ so by Lemma 4.4.6 $q_{j+r-1} \leq q_{j+r}$. Now $q_{j-1} \leq q_j = q_{j+r} \leq q_{j+r-1} = q_{j-1}$. So $q_1 = q_2 = \cdots = q_m$.

If $a_{p_m} \neq a_{p_i}$ for some $m < i \leq r$ then for the smallest i, $a_{p_{m-1}} = a_{p_i}$ and $a_{p_m} \neq a_{p_{m+1}}$ since $s_{i_{m-1}}, s_{i_{m+1}} \in A^{(k)}$. Then $a_{p_{m-1+r}}^{(q_{m-1+r})}, a_{p_{m+r}}^{(q_{m+r})}, a_{p_{i+r}}^{(q_{i+r})}$ yields the palindrome $a_{p_{m-1+r}}, a_{p_{m+r}}, a_{p_{i+r}}$ in the underlying sequence of S' that is a palindrome in A which is a contradiction.

So we may suppose that $a_{p_m} = a_{p_{m+1}} = \cdots = a_{p_r}$ and $q_m < q_{m+1} <$

$\cdots < q_r$. Since S' is a repetition it follows that $a_{p_{m+r}} = a_{p_{m+r+1}} = \cdots = a_{p_{2r}}$. So we get that $q_{r+1} = q_1 = q_m < q_r$ and so $a_{p_{r+1}} \neq a_{p_r} = a_{p_m}$. Since $a_{p_m} = a_{p_r}$ we must have $a_{p_{m-1}} = a_{p_{r+1}}$, since they come from the same underlying symbol of A. Similarly $p_{m-2} = p_{r+2}, p_{m-3} = p_{r+3}, \ldots, p_1 = p_{r+m-1}$. Thus $a_{p_r}, a_{p_{r+1}}, a_{p_{r+2}}, \ldots, a_{p_{r+m-1}}, a_{p_{r+m}}$ is a palindrome in A because for $0 \leq j < m$ we have $a_{p_{r+j}} = a_{p_{m-j}} = a_{p_{r+m-j}}$.

□

This result directly yields a new proof of Theorem 2.3.6.

Corollary 4.4.8. *$\pi'(T_{k,h}) \leq 4k$ for all k, h and $\pi(T_{k,h}) \leq 3k$ for $h \leq 5$.*

Proof. A repetition-free and palindrome-free sequence of arbitrary length can be created using four symbols thus $A^{(k)}$ can be created using $4k$ symbols. $A^{(k)}$ is k-special so the coloring derived from $A^{(k)}$ is repetition-free. The result then follows from Theorem 2.1.5 and the observation that the sequence 12312 is repetition-free and palindrome-free. □

Observe that we only used the property that A is palindrome-free in case 4 of the proof of Theorem 4.4.7. This observation leads us to believe that this condition may be relaxed in some fashion, a possibility which we will discuss in Section 4.6.

4.5 2-special sequences with few symbols

Lemma 4.5.1. *Let $S = s_1, s_2, \ldots$ be a sequence. If $s_i = s_j$ and $1 \leq |i - j| \leq 2k - 1$, then S is not k-special.*

Proof. Let $S = s_1, s_2, \ldots$ be a sequence, $s_i = s_j$, and without loss of generality $i < j$. If $j \leq i + k$ then the sequence of indices j, i obeys the four conditions from Definition 4.2.1 with $m = 2$. However, if $j > i + k$, then the sequence $j, i + k - 1, i, i + k - 1$ obeys the four conditions with $m = 3$. In either case we see that S is not k-special. □

Corollary 4.5.2. *Let $S = s_1, s_2, \ldots$ be a sequence. If $s_i = s_j$ and $1 \leq |i - j| \leq 3$, then S is not 2-special.*

Theorem 4.5.3. *The longest 2-special sequence on three symbols is 1,2,3.*

Proof. By Corollary 4.5.2 the sequences 1,1 and 1,2,1 are not 2-special, so up to changing of symbols 1,2,3 is the only 2-special sequence of length 3. If a 2-special sequence on three symbols had length four the derived edge-coloring would be a repetition-free edge-coloring of $T_{2,2}$ which contradicts the fact that $\pi'(T_{2,2}) = 4$.

□

Theorem 4.5.4. *The longest 2-special sequence on four symbols is 1,2,3,4,1.*

Proof. By Corollary 4.5.2 and Theorem 4.5.3 we may start with 1,2,3,4. We proceed exhaustively. By Corollary 4.5.2 the next entry can not be any of the previous three, so we exclude symbols 2, 3, and 4 which means the next entry must be 1. So far we have the sequence 1,2,3,4,1 which can be verified to be 2-special. By Corollary 4.5.2 the next entry can not be any of the previous three, so we exclude symbols 1, 3, and 4 which means the next entry must be 2. However as mentioned previously in Example 4.3.3 the sequence 1,2,3,4,1,2 is not 2-special and so the result follows. □

Once we start using five or more symbols it begins to get harder to exhaustively find the longest sequence by hand. We obtained the results in the next two theorems using Algorithm 9. The running time for five and six symbols was approximately 5 seconds and 7 minutes, respectively.

Theorem 4.5.5. *The longest* 2*-special sequence on five symbols has length 8. Up to isomorphism there are exactly two sequences of this length: 1, 2, 3, 4, 5, 1, 3, 2 and 1, 2, 3, 4, 5, 1, 2, 3.*

Theorem 4.5.6. *The longest* 2*-special sequence on six symbols has length 73. Up to isomorphism there are exactly two sequences of this length:*
1, 2, 3, 4, 5, 6, 3, 2, 1, 5, 6, 4, 1, 2, 3, 6, 5, 1, 4, 6, 3, 2, 1, 4, 5, 2, 3, 6, 4, 1, 3, 2, 5, 4, 1, 6, 5, 2, 3, 1, 4, 5, 6, 1, 3, 2, 5, 4, 3, 1, 6, 5, 4, 3, 2, 5, 6, 1, 3, 4, 5, 2, 3, 1, 6, 5, 2, 4, 6, 1, 3, 5, 6.

1, 2, 3, 4, 5, 1, 3, 2, 6, 5, 1, 4, 6, 2, 3, 1, 5, 6, 4, 1, 3, 2, 6, 4, 5, 2, 3, 1, 4, 6, 3, 2, 5, 4, 6, 1, 5, 2, 3, 6, 4, 5, 1, 6, 3, 2, 5, 4, 3, 6, 1, 5, 4, 3, 2, 5, 1, 6, 3, 4, 5, 2, 3, 6, 1, 5, 2, 4, 1, 6, 3, 5, 1.

After running Algorithm 9 in Appendix with $k = 2$ and 7 colors for approximately an hour the longest 2-special sequence we found had length 458. 1, 2, 3, 4, 7, 6, 5, 4, 3, 2, 7, 6, 3, 1, 5, 7, 3, 2, 6, 7, 5, 4, 6, 3, 1, 7, 6, 5, 2, 3, 7, 4, 2, 6, 1, 4, 5, 7, 6, 3, 5, 4, 1, 7, 6, 2, 4, 7, 5, 3, 6, 4, 2, 3, 7, 1, 6, 5, 7, 4, 3, 6, 5, 7, 2, 6, 4, 1, 7, 6, 3, 2, 7, 5, 4, 6, 7, 3, 1, 5, 7, 6, 4, 5, 3, 2, 6, 4, 5, 7, 6, 2, 3, 5, 7, 4, 3, 2, 1, 7, 5, 4, 6, 7, 3, 1, 5, 2, 7, 6, 5, 4, 3, 7, 6, 5, 2, 7, 4, 1, 6, 3, 7, 5, 1, 4, 7, 6, 2, 5, 3, 7, 6, 2, 4, 7, 5, 1, 6, 4, 3, 2, 6, 7, 5, 4, 3, 6, 1, 4, 7, 2, 3, 5, 7, 6, 4, 5, 3, 2, 7, 6, 3, 1, 5, 7, 6, 3, 2, 7, 5, 4, 6, 7, 3, 1, 6, 5, 2, 3, 6, 7, 4, 5, 6, 3, 2, 5, 7, 1, 3, 5, 4, 6, 7, 3, 2, 6, 4, 5, 7, 6, 3, 1, 7, 5, 4, 2, 7, 1, 6, 4, 5, 3, 6, 7, 2, 4, 3, 6, 5, 1, 7, 6, 3, 4, 7, 1, 5, 6, 7, 3, 2, 5, 7, 6, 4, 5, 2, 1, 7, 6, 2, 3, 5, 7, 6, 2, 1, 7, 5, 4, 6, 7, 3, 2, 6, 5, 1, 3, 6, 7, 4, 5, 6, 3, 2, 4, 7, 5, 1, 6, 4, 7, 5, 3, 2, 6, 5, 7, 4, 6, 3, 1, 7, 4, 6, 5, 7, 3, 2, 6, 7, 5, 1, 4, 2, 7, 6, 5, 3, 7, 4, 1, 6, 7, 3, 2, 5, 7, 4, 1, 5, 3, 6, 4, 5, 7, 2, 6, 5, 4, 3, 2, 7, 6, 5, 2, 1, 4, 7, 6, 3, 4, 5, 2, 7, 6, 4, 3, 7, 2, 5, 6, 7, 4, 1, 6, 5, 3, 7, 6, 4, 2, 7, 5, 1, 4, 7, 6, 2, 4, 3, 5, 6, 7, 3, 4, 1, 7, 5, 2, 4, 7, 6, 3, 5, 7, 4, 2, 5, 6, 1, 4, 5, 3, 7, 6, 4, 2, 7, 1, 5, 4, 7, 6, 3, 5, 7, 4, 1, 3, 7, 6, 5, 3, 2, 4, 7, 6, 5, 4, 3, 1, 7, 4, 5, 6, 7, 3, 2, 6, 5, 1, 4, 7, 6, 5, 3, 7, 4, 2, 3, 1, 6, 7, 4, 5, 6, 2, 3, 7, 6, 5, 4, 7, 3, 2, 6, 7, 5, 1, 3, 7, 6, 4, 3, 5

The coloring derived from this sequence will give a repetition-free edge-coloring of $T_{2,h}$ where $h = 458/2 = 229$. This sequence is far more compact than the color sequences we used in Chapter 3. To put this into perspective the color sequence in Example 3.2.6 had 2046 entries (one for each edge) but only colored $T_{2,10}$.

We will now update Table 2.1 to reflect all of the results from Chapter 3 and Chapter 4. Improvements are bold and shown in red.

k/h	1	2	3	4,5	≤ 10	≤ 36	≤ 229	≥ 230
1	1	2	2	3	3	3	3	3
2	2	4	**4**	**5**	**5**	**5,6**	**5,7**	**5**,8
3	3	5	5, 8	5, 9	5, 12	5, 12	5, 12	5, 12
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	k	$\lfloor 1.5k \rfloor + 1$	$1.61k$, $\lfloor 2.5k + 1 \rfloor$	$1.61k$, $3k$	$1.61k$, $4k$	$1.61k$, $4k$	$1.61k$, $4k$	$1.61k$, $4k$

Table 4.1

4.6 Future directions

We hope to weaken the hypothesis of Theorem 4.4.7. If we are successful this could lead to an improvement on the upper bound of $4k$ from Corollary 4.4.8.

The repetition-free condition is obviously necessary as any repetition in A immediately implies that $A^{(k)}$ is not k-special. However, the repetition-free condition by itself is not sufficient because Algorithm 5 can be used to verify that for the repetition-free sequence

$A = 1, 2, 3, 1, 2, 1, 3, 1, 2, 3, 2, 1, 3, 2, 3, 1, 3, 2, 1, 2, 3, 1, 2$ from Example 2.1.3 we get that $A^{(2)}$ is not 2-special. We conjecture that in fact $A^{(2)}$ being 2-special is sufficient.

Conjecture 4.6.1. *If $A^{(2)}$ is 2-special, then $A^{(k)}$ is k-special.*

This conjecture leads to the following more general question:

Question 4.6.2. *For which $k, l \geq 2$ does $A^{(l)}$ being l-special imply that $A^{(k)}$ is k-special?*

Answering the following related question would shed light on the case when k in Question 4.6.2 is a multiple of l, and could in itself also lead to improved bounds for $\pi'(T_{k,h})$.

Question 4.6.3. *For which $k, l \geq 2$ does A being l-special imply that the k-extension of A is (kl)-special?*

Appendix A

Computer programs

In this appendix we include some computer programs that were helpful in the investigation of repetition-free edge-colorings and k-special sequences. We begin by explaining some general notation and will be using pseudo code for all the algorithms. In Section A.2 we give an algorithm for verifying if an edge-coloring of a k-ary tree is repetition free. In Section A.3 are algorithms for verifying if a sequence is k-special. Finally in Section A.4 we give an algorithm that generates a list of maximal k-special sequences for a given number of colors.

A.1 General conventions

For our algorithms we are working with a labeled $T_{k,h}$ with $n = 1 + k + k^2 + \cdots + k^h$ vertices that has a proper edge-coloring c with color sequence $C = c(e_1), c(e_2), \ldots c(e_{n-1})$. For our inputs we will use an integer k to represent k-ary. C will be a list of length $n - 1$ and in position i will be the color of edge e_i. Vertices are integer values based on their labels, so they go from 1 to n and we usually use v or u to represent them. We will use *rep* and *repfree* as boolean values. S is a repetition-free sequence on symbols $\{1, 2, \ldots m\}$ of length $2h$ that is entered as a list. We view the symbols as colors (and refer to them as such hereafter) since the edge-coloring derived from S produces an m-edge-coloring of $T_{k,h}$.

When we say **return** the algorithm terminates and outputs the specified value. Let L be a list. If we write $L[i]$ we mean the i^{th} element of L. If the elements of L are lists and we write $L[i][j]$ then we mean the j^{th} element of $L[i]$. The command L append i will add i to the end of list L and delete $L[i]$ will delete element i from L. When we use delete it will shift the entries with inices $i+1, i+2, \ldots$ so that $L[i+1]$ will now be in position i, $L[i+2]$ will be in position $i+1$, and so on. Length(L) is the length of L.

The top line of each algorithm will contain the name of the algorithm, followed by a list of input variables, and finally an $\rightarrow$ followed by the data-type that is returned by the algorithm. Our input data as such will not be modified by the algorithms, except for the queue Q in Algorithms 6-8.

A.2 Verifying repetition-free colorings

Algorithm 1 determines the color of the edge e_{v-1} above vertex v for the given color sequence C.

Algorithm 1 COLOR(v,C) $\rightarrow$ Integer

```
return C[v − 1]
```

Algorithm 2 determines the parent of vertex v. If v is the root (which does not have a parent), then the value will be zero.

Algorithm 2 PARENT(v,k) → Integer

```
return ⌊(v + k − 2)/k⌋
```

Algorithm 3 will take as its input the color sequence of an edge-coloring that we assume to be proper and return TRUE if the coloring is repetition-free. When programmed in Python it takes approximately two seconds to verify that the coloring from Example 3.2.6 is repetition-free.

Algorithm 3 REPETITIONFREE(n,k,C) → Boolean

```
repfree ← TRUE
for v = k + 2 to n do    ▷ We do not need to check the vertices in level 1
    w ← PARENT(v,k)
    u ← PARENT(w,k)
    while repfree=TRUE and u > 0 do
        if COLOR(w,C) ≠ COLOR(v,C) then
            if VERIFY(u,v,k,C)=TRUE then
                repfree ← FALSE
            end if
        end if
        w ← PARENT(w,k)
        u ← PARENT(u,k)
    end while
end for
return repfree    ▷ True means repetition-free
```

Algorithm 3 checks if a proper edge-coloring C of $T_{k,h}$ contains a repetition on some path $v_0, v_1, \ldots v_{2r}$ for some $r \geq 2$. Observe that if C contains such a repetition, then we can assume that it contains a repetition in which v_r is an ancestor of v_0. Algorithm 3 finds such a repetition by using the subroutine VERIFY(u,v,k,C) to verify if there is such a repetition in which $v_0 = v$ and $v_r = u$ for a specific ancestor u of v. It suffices to check all u on the path from v to the root. Note that we will only call VERIFY on those u such that the color on the edge $vp(v)$ is different from the color on the edge wu where $u = p(w)$ and w is an ancestor of v, since otherwise VERIFY(u,v,k,C) would falsely return true when the coloring on the u, v-path is a palindrome.

Algorithm 4 VERIFY(u,v,k,C) → Boolean

```
x ← u
y ← v
rep ← TRUE
while rep =TRUE and y > u do            ▷ If y = u we have a repetition
    rep ← FALSE
    if Color(y,C)=Color(x,C) then
        x ← Parent(x,k)
        rep ← TRUE
    else
        for i = 1 to k do
            z ← k(x − 1) + 1 + i                  ▷ The children of x
            if Color(z,C)=Color(y,C) then
                x ← z
                rep ← TRUE
            end if
        end for
    end if
    y ← Parent(y,k)
end while
return rep
```

Given two vertices u, v such that u is an ancestor of v and a proper coloring C, Algorithm 4 checks if there is a path $v_0, v_1, \ldots v_{2r}$ with $v = v_0$ and $u = v_r$ whose color pattern is a repetition. The output is TRUE if there is such a repetition and FALSE otherwise.

We initially assume that $x = u$ and $y = v$. We compare the colors of the edge above y and the edges containing x. If any of these colors are equal then we could potentially have a repetition so we move x and y to the vertices incident with the edges whose colors are equal. These are unique since the coloring is proper. If there are no edges with the same colors we do not have a repetition and we stop the algorithm. We continue this process until we either cannot move (there are no colors in common) or y has reached the position of u in which case we have found a repetition.

This algorithm should not be run by itself, it may falsely return true if the u, v-path P has a color pattern that is a palindrome.

A.3 Verifying k-special sequences

The first algorithm in this section returns the answer TRUE precisely when the given input sequence S is k-special.

Algorithm 5 KSPECIAL(k,S) $\rightarrow$ Boolean

```
for u = 1 to length(S) do
    for v = u + 1 to length(S) do
        if UVSPECIAL(k,u,v,S)=TRUE then
            return FALSE                          ▷ There is a repetition
        end if
    end for
end for
return TRUE
```

Algorithm 5 checks if an array S has a sequence of indices $i_1, i_2 \ldots i_{2r}$ for some r that satisfies the four conditions of Definition 4.2.1. If for such a sequence we have $m > r$, then $i_1 > i_2 > \cdots > i_{r+1}$ and we let $v = i_1$ and $u = i_{r+1}$. If on the other hand $m \leq r$, then $i_{2r} > i_{2r-1} > \cdots > i_r$ and we let $v = i_{2r}$ and $u = i_r$. In either case we can find an endpoint v of our sequence such that the block of indices from v to the corresponding midpoint u with the same color is monotone. The subroutine uvSpecial(k, u, v, S) described in Algorithm 6 returns TRUE precisely when such a repetition exists for a fixed pair of indices u, v with $u < v$.

Algorithm 6 uvSpecial(k,u,v,S) → Boolean

```
if S[u] ≠ S[v] then
    return FALSE
end if
if v − u ≤ 2k − 1 then                              ▷ Repetition by Lemma 4.5.1
    return TRUE
end if
Q ← [[u, v, left, FALSE]]
while Q ≠ ∅ do
    x ← Q[1][1]
    y ← Q[1][2]
    dir ← Q[1][3]
    long ← Q[1][4]
    if y ≤ u + k and (y, dir, long) ≠ (u + k, right, TRUE) then
        return TRUE
    end if
    if dir=right then
        Right(Q,S,x,y,long)
    else
        Left(Q,S,x,y,long)
    end if
    delete Q[1]                                     ▷ Remove the head of Q
end while
return FALSE
```

We want to find a subsequence of S with indices $i_1 > i_2 \cdots > i_r > \cdots > i_m < i_{m+1} \cdots < i_{2r}$ with $u = i_1$ and $v = i_{r+1}$ such that $S[i_l] = S[i_{l+r}]$ for $1 \leq l \leq r$, $|i_j - i_{j+1}| \leq k$ for $1 \leq j \leq 2r$, and either $i_{m+1} < i_m + k$ or $i_{m-1} < i_m + k$. We will do so by creating a queue Q of quadruples $[x, y, dir, long]$ where x, y are indices, dir is a direction that is $left$ or $right$, and $long$ is a boolean variable that is TRUE when we make a step of size k.

We require for each quadruple $[x, y, dir, long]$ that there is a sequence of indices $i_1, \ldots, i_l$ and $i_{r+1}, \ldots i_{r+l}$ with the following properties: $S[i_j] = S[i_{r+j}]$ for $1 \leq j \leq l$, $|i_j - i_{j+1}| \leq k$ for $j \in \{1, 2, .., l-1, r+1, r+2, r+l-1\}$, $v = i_1 > i_2 \cdots > i_l = y$ and either $dir{=}left$ and $u = i_{r+1} > \cdots > i_{r+l} = x$ or $dir{=}right$ and $u = i_{r+1} > \cdots > i_m < \ldots i_{r+l} = x$ with $i_{m+1} < i_m + k$ or $i_{m-1} < i_m + k$. Moreover if $long$=FALSE and $dir{=}left$ we require that $i_{r+l-1} < i_{r+l} + k$ whereas if $long$=TRUE and $dir{=}right$ then we are in the situation where $i_m = i_{r+1} = i_{r+2} - k$ indicating that we started the second half of our repetition with a long step to the right. These conditions are satisfied by $[u, v, left$,FALSE] and we use this to initialize Q.

Each subsequence will be half of our repetition, we may join these two together when the last index of the first sequence y is within k of the first index of the second sequence u. If $y \leq u + k$ then $|y - u| \leq k$ and we are able to join our two sequences of indices into a single sequence $i_1, i_2, \ldots, i_{2r}$ by letting $r = l$. This sequence forms a repetition with $i_1 = v > i_2 \cdots > i_l = y > u = i_{r+1} > \cdots > i_m < i_{m+1} \cdots < i_{2r}$ and $|i_j - i_{j+1}| \leq k$ (line 12). We may assume that $i_{m+1} < i_m + k$ or $i_{m-1} < i_m + k$ as repetitions in reverse order are still repetitions. Thus this subsequence satisfies the four conditions from Definition 4.2.1 so S is not k-special. If such a subsequence can not be found then the sequence is k-special and we return FALSE (line 13).

In lines 15-19, Algorithm 6 now tries to extend the two given sequences based on whether we moved to the left or right to reach x using Algorithm 7 or Algorithm 8 accordingly.

Algorithm 7 RIGHT(Q,S,x,y,$long$)

for $b = 1$ **to** k **do**
    for $a = 1$ **to** k **do**
        if $S[x + a] = S[y - b]$ **then**
            Q append $[x + a, y - b, right, long]$
        end if
    end for
end for

When *dir=right* the only way to extend the current sequence is to find i_{l+1} with $i_l > i_{l+1} \geq i_l - k$, and i_{r+l+1} with $i_{l+r} < i_{l+r+1} \leq i_{l+r} + k$ and $S[i_{l+1}] = S[i_{r+l+1}]$. Recall that $u = i_{r+1} > \cdots > i_m < \ldots i_{r+l} = x$ so in order to satisfy the four conditions from Definition 4.2.1 when we join our two subsequences we need the indices to continue moving in the same direction. The boolean value *long* is needed if our first step from u_{r+1} is k to the right so $i_{r+2} = i_{r+1} + k$. If this happens then our last step from y can not also be a step of size k as $u = i_m$ in this case and we would have $i_m + k = i_{m+1} = y$ and $i_m + k = i_{m-1} = i_{r+2}$ which does not satisfy the last condition from Definition 4.2.1.

Algorithm 8 LEFT(*Q,S,x,y,long*)

```
1: for b = 1 to k do
2:     for a = 1 to k − 1 do
3:         if S[x − a] = S[y − b] then
4:             Q append [x − a, y − b, left, FALSE]
5:         end if
6:         if S[x + a] = S[y − b] then
7:             Q append [x + a, y − b, right, FALSE]
8:         end if
9:     end for
10:    if S[x − k] = S[y − b] then
11:        Q append [x − k, y − b, left, TRUE]
12:    end if
13:    if S[x + k] = S[y − b] and long = FALSE then
14:        Q append [x + k, y − b, right, (x = u)]
15:    end if
16: end for
```

When dir=$left$ we can extend $v > \cdots > y$ to the left and extend $u > \cdots > x$ to either the left or the right. In either case we must find i_{l+1} and i_{r+l+1} with $0 < b = i_l - i_{l-1} \leq k$ (line 1), $0 < a = |i_{r+l+1} - i_{r+l}| \leq k$, and $S[i_{l+1}] = S[i_{r+l+1}]$ (line 3,6,10,13). If $i_{r+l} > i_{r+l+1} \geq i_{r+l} - k$ then we can move to the left and *long*=FALSE (line 4). If $i_{r+l+1} = i_{r+l} - k$ then we can also move left but we must make *long*=TRUE (line 11). If $i_{r+l} < i_{r+l+1} \leq i_{r+l} + k$ then we want to turn around and switch directions, that is $m = r + l$ so the last condition of Definition 4.2.1 is satisfied. We can do so if either $i_{m+1} \leq i_m + k - 1$ (line 6) or if $i_{m-1} \leq i_m + k - 1$, that is *long*=FALSE (line 13). Observe that if $x = u$ (that is we turned around in our very first step), then we will set *long*=TRUE if our first step was of length k (line 14), so that we can correctly address this case in line 13 of Algorithm 6.

A.4 Generating k-special sequences

Our last algorithm generates a list L of all maximal k-special sequences on $\{1, 2, \ldots, m\}$ that start with $1, 2, 3, \ldots, 2k$. The elements of L are ordered by their length. Observe that by Lemma 4.5.1 the first $2k$ elements of any k-special sequence must be distinct and our choice of starting with $1, 2, \ldots, 2k$ filters out many isomorphic sequences. This algorithm can easily be modified to find all maximal k-special extensions of an arbitrary sequence S, by making S part of the input and replacing the first line with $Q \leftarrow [S]$.

Algorithm 9 EXHAUSTIVE(k,m) → List

```
Q ← [[1, 2, 3, ... 2k]]
L ← [ ]
while Q ≠ ∅ do
    A ← Q[1]                                   ▷ First entry of Q.
    Q_1 ←[ ]                                   ▷ List of possible entries to add to Q.
    bad ←[ ]                                   ▷ Resets the bad colors.
    Q ← delete Q[1]                            ▷ Deletes A from Q
    for j = 1 to 2k − 1 do   ▷ Creates a list of the last 2k − 1 entries of A.
        bad append A[length(A) − j + 1]
    end for
    D ← [1, 2, ... m] − bad                    ▷ Colors to try for an extension.
    for j = 1 to length(D) do
        B ← A append D[j]
        if kSPECIAL(k,B)=TRUE: then            ▷ A can be extended.
            Q_1 append B
        end if
    end for
    if length(Q_1) = 0 then                    ▷ A had no extensions.
        L append A
    else:
        Q ← Q ∪ Q_1                            ▷ Adds the new possibilities to Q
    end if
end while
return L
```

The algorithm works by maintaining a queue Q of lists that are k-special and can possibly be extended. The algorithm takes the first sequence A of Q and tries to extend it. From Lemma 4.5.1 we know the last $2k-1$ entries of A cause repetitions so we remove these colors $\{1, 2, \ldots m\}$, we call the resulting list D. Next we try each entry of D at the end of A and use Algorithm 5 to check if the new sequence is k-special, if it is we add this new sequence to Q_1. Once we have added all of the extensions of A to Q_1 we delete A from Q and add all the entries of Q_1 to Q. If there were no extensions of A, Q_1 will be empty and we know that A is maximal and can not be extended, we then delete A from Q.

When we add entries to our queue we add them at the tail and remove the head which creates a breadth-first search. This is preferable as the outputs will be somewhat ordered (we do all of the extensions of a given level before proceeding to the next), however the largest entry will be at the end. If we changed A to be the tail of Q we could change to a depth-first search. This approach would be advantageous when searching for a k-special extension that is as long as possible in a limited amount of time.

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