Towards an Algebra of Normal Complexes

A Thesis submitted to the faculty of
San Francisco State University
In partial satisfaction of the
requirements for
the Degree

Masters of Arts
in
Mathematics

by
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San Francisco, California
December 2023
Abstract

The polytope algebra is well-studied and features several interesting properties. One of these is the nilpotency $(\lfloor f_P \rfloor - 1)^{\dim P} = 0$. Normal complexes, first described by Nathanson and Ross in 2021, are polytopal complexes obtained by truncating a fan $\Sigma$. First, we define an analogue to the polytope algebra for normal complexes. Then, we prove that this nilpotency $(\lfloor f_N \rfloor - 1)^{\dim \Sigma} = 0$ holds in the case one-dimensional fans.
Acknowledgments

I want to thank the BAMM! program for supporting me financially during my Master’s program. I want to offer specific thanks to the BAMM! director at SFSU, Kim Seashore, for her guidance and for being an advocate for me.

I wish I could thank each and every one of my peers here by name, and for their specific contributions my academic career, but that would surely take much more space than I have here. I would specifically like to thank my research partner Ian Wallace, for taking this journey with me, and offering invaluable support in research, as well as Christina Nguyen, Patrick O’Melveny, Anika O’Donnell, and Jonathan Farley for their friendship and emotional support.

I would like to extend a thanks to my committee members, Matthias Beck and Emily Clader: Dr. Beck for holding me to the highest standards and pushing me to be the best student I can be, and Dr. Clader for fostering a collaborative and supportive mathematical community that has allowed me to thrive, and being the faculty advisor for Mathematistas. The department would not be what it is without its faculty, and I want to thank all of my professors for facilitating my success.

Finally, and most importantly, I would like to thank Dusty Ross, for taking me on as an advisee, for his infinite enthusiasm throughout our work together, and for encouraging me to write this strange and creative thesis.
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Chapter 1

Introduction

Like all artists, mathematicians blend inspiration from previous works, both new and old, to try to create something new. In this paper, we look at normal complexes, a recent construction first described in 2021 by Nathanson and Ross [3]. A normal complex is an orthogonal truncation of a fan. In this paper, we explore a new property of normal complexes by placing them into an algebra. Our inspiration can be traced back to similar results in the polytope algebra, which were demonstrated by both McMullen [2] and Pukhlikov and Khovanskiĭ [5]. The algebra for normal complexes holds the potential to show even more connections between normal complexes and the broader world of algebraic combinatorics.

In the first section of the first chapter of this thesis, we cover some of the mathematical structures that are used throughout the paper. We introduce the polytope algebra, from which we take inspiration, by defining polytopes and some of their features. Then, we describe the operations that make up the polytope algebra. We end the first chapter with
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a proof of the equation \((f_P - 1)^2 = 0\), where \([f_P]\) is an element of the one-dimensional polytope algebra.

Next, we enter the world of normal complexes. We define what they are, and begin following the lead of polytopes by placing normal complexes in their own algebra.

The central result of the thesis can be found in the chapter “A Nilpotency in the Algebra for Normal Complexes,” where we discuss why \((N - 1)^2\) holds for an element \([N]\) of an algebra for one-dimensional normal complexes. First, we cover an example that shows how the general-case proof will work. Next, we actually prove the general case.

Finally, we orient ourselves in the larger field of algebraic combinatorics by discussing how further work can be done in proving things about the algebra for normal complexes.

Throughout, you will find images that present the concepts of the paper in metaphorical ways that are not necessarily mathematically rigorous. This is because mathematics is never truly independent from the people who write it. While much of the text seeks to exposit the story of a proof, these panels share the story of how identity, disability, and artistic expression influence the joyful and stressful ways in which we interact with mathematics.
What’s Going on Here?

A casual reader of this thesis may, at some points, prefer a less technical, more “hand-wavey” bird’s-eye perspective on what is being discussed. To more clearly delineate these broad, imprecise descriptions from more rigorous explanations, they will be wrapped in separate purple boxes. The intended audience for these sections is someone like the author’s parents, readers who are inexperienced in formal mathematics but are nevertheless invested in understanding the work. However, mathematicians may still find this exposition enlightening for a more general understanding of the overarching ideas in this paper.
Chapter 2

Background

2.1 The Rules of the Game

Some people think I’m a pretender, like I became a combinatorialist to count made-up genders.

One question this thesis will not seek to answer is “Why do we do mathematics?” Because you are reading this thesis, hopefully voluntarily, you should have your own reasoning for embarking on this journey of mathematical discovery.

The first questions this paper will answer is “Why do mathematicians organize mathematical objects into structures?” and the answer here is “To make strange and new objects into something familiar.” The normal complex is a very recent mathematical construct, and we have only begun to understand what its properties are and how they can be used. Polytopes, on the other hand, are well-studied. By putting both objects into similarly orga-
Figure 2.1: We would like to get to know normal complexes (right) as well as we know polytopes.

nized structures, we can see what properties they have in common. In this way, we get to know normal complexes, and understand how their traits might be similar to those of the polytopes we are already acquainted with.

Algebraic structures consist of elements, along with different ways to combine these elements, called operations. Mathematical structures must abide by some predefined properties in order to make them easier to work with. Mathematics is like a sport, and these properties ensure that math follows some predictable patterns, while allowing for the flexibility and excitement of gameplay.

A few of our favorite “games” are rings, fields, and vector spaces, and they will likely be familiar friends to the reader with a semester or more of abstract algebra under their
CHAPTER 2. BACKGROUND

belt. Here, we assume knowledge of their definitions. However, some readers may have never heard of them before. Those readers are referred to the “What’s Going on Here?” section at the end of the chapter to provide an overview of what is being discussed. For the experienced mathematician ready to understand the details of the full story, we begin with a quick reminder about what it means to be commutative.

Definition 2.1.1. A commutative operation $*$ on a set $A$ is any operation such that, for all $a, b \in A$, $a * b = b * a$.

In this paper, we will only use operations that are commutative. The major structure we will work with is the $\mathbb{R}$-algebra.

Definition 2.1.2. An $\mathbb{R}$-algebra is a vector space $V$ over $\mathbb{R}$, along with a multiplicative operation $\times$ on $V$ satisfying:

1. $(x + y) \times z = x \times z + y \times z$ (left distribution)

2. $z \times (x + y) = z \times x + z \times y$ (right distribution)

3. $(ax) \times (by) = (ab)(x \times y)$ (compatibility with scalars)

for all $x, y, z \in V, a, b \in \mathbb{R}$.

Note that when $\times$ is commutative, the first and second properties are equivalent.

In other words, an $\mathbb{R}$-algebra consists of two sets, $\mathbb{R}$ and something else, and three operations: addition, multiplication, and scalar multiplication.
Example 2.1.1. Consider the real polynomials with one variable, $\mathbb{R}[x]$. Let $p(x)$, $q(x)$, and $r(x) \in \mathbb{R}[x]$. Since polynomial multiplication is distributive, we know that

1. $(p(x) + q(x)) \times r(x) = (p(x) \times r(x)) + (q(x) \times r(x))$, and 

2. $r(x) \times (p(x) + q(x)) = (r(x) \times p(x)) + (r(x) \times q(x))$.

We also know that polynomial multiplication commutes with scalar multiplication, so we also know that

3. $(a \cdot p(x)) \times (b \cdot q(x)) = (ab) \cdot (p(x) \times q(x))$.

Therefore, we conclude that the real polynomials with one variable form an $\mathbb{R}$-algebra.

Later in this paper, we will form quotient algebras using ideals, so we will define these terms here.

Definition 2.1.3. An ideal is a sub-algebra $\mathcal{I} \subseteq A$ such that for all $d \in \mathcal{I}$, $a \in A$, it holds that $ad \in \mathcal{I}$.

In other words, the elements of the ideal are infectious, so multiplying any element of the algebra by something in the ideal results in an element of the ideal.

Definition 2.1.4. A quotient algebra, $A/\mathcal{I}$ for an algebra $A$ and an ideal $\mathcal{I}$ is the set of cosets, defined

$$[a] := a + \mathcal{I}$$
with the operations

\[ [a] + [b] = [a + b], \]

\[ [a] \times [b] = [a \times b], \] and

\[ r[a] = [ra]. \]

for \( a, b \in A \) and \( r \in \mathbb{R} \), such that the cosets, along with these operations, satisfy the properties of an \( \mathbb{R} \)-algebra.

**Example 2.1.2.** Consider the algebra of polynomials in two variable, \( \mathbb{R}[x, y] \). We can take the ideal generated by \( xy \),

\[ \mathcal{I} = \langle xy \rangle = \{ xyf \mid f \in \mathbb{R}[x, y] \}. \]

This is an ideal which contains any multiple of \( xy \). When we take the quotient \( \mathbb{R}[x, y]/\mathcal{I} \), any polynomial that is a multiple of \( xy \) becomes 0. The elements of the quotient algebra are
linear combinations of powers of $x$, powers of $y$, and constants.

At last, we have the rules of the game. In order to start playing, we next introduce our first group of players. They will be indicator functions on polytopes, which we will explore in the next section. Later, we will see how we can play the same game with a new group of players, called normal complexes.

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**What’s Going on Here?**

One of the coolest parts of mathematics is the ability to do operations like addition and multiplication on different kinds of objects. Polytopes and normal complexes are going to get a lot more complicated than the single values you might be used to doing math with. In order for these operations to work, we have to do some serious work ensuring certain properties are obeyed, so nothing unexpected happens. It is possible for someone to understand the basics of the algebra on normal complexes without knowing the nitty-gritty detail. To the advanced reader, these definitions provide a nice scaffolding as we proceed to define the polytope algebra and the algebra of normal complexes.
2.2 The Polytope Algebra

I’ve heard of doing algebra, but I’ve never heard of “an” algebra.

In order to construct an algebra for normal complexes, we must first understand the polytope algebra. The construction of the algebra for normal complexes draws inspiration from the polytope algebra, which was first described by McMullen in 1989 [2]. However, the characterization of the polytope algebra used in the following chapters was developed by Pukhlikov and Khovanskii in 1992 [5]. A result of Groemer shows that the algebras described in both of these papers are in fact the same [1].

Polytopes can be defined differently depending on what you’d like to use them for, and in this paper we will use the definition from halfspaces. This is made easier with some notion of orthogonality. In order to define orthogonality, we use an inner product, $\ast$, on $\mathbb{R}^d$. In general, we will assume the inner product is the dot product, but any inner product will do.

Definition 2.2.1. A hyperplane $H_{a,b}$ is a subset of $\mathbb{R}^d$ of the form

$$\{x \in \mathbb{R}^d \mid a \ast x = b\},$$

where $a \neq 0$ is some vector in $\mathbb{R}^d$, and $b \in \mathbb{R}$.

Each hyperplane results in two potential halfspaces, one on either side. We denote the halfspaces associated with $H_{a,b}$ as $H_{a,b}^+$ and $H_{a,b}^-.$

Definition 2.2.2. A halfspace, denoted $H_{a,b}^+$ or $H_{a,b}^-$ is a subset of $\mathbb{R}^d$ of the form

$$H_{a,b}^+ := \{x \in \mathbb{R}^d \mid a \ast x \geq b\}.$$
\[ \mathcal{H}_{a,b} := \{ x \in \mathbb{R}^d \mid a \cdot x \leq b \}. \]

**What’s Going on Here?**

A hyperplane is an object of dimension \( d - 1 \) in an \( d \)-dimensional space. So in the plane, a hyperplane is a line, in 3-space a hyperplane is a plane, and so on for any number of dimensions. A hyperplane cuts our space in “half” (even though the space is infinite), and we can refer to everything on one side of the hyperplane as a halfspace.

Now, we have all the background we need to define a polytope.

**Definition 2.2.3.** A polytope \( P \) is a subset of \( \mathbb{R}^d \) that is a bounded intersection of finitely many halfspaces of \( \mathbb{R}^d \). An example of a polytope can be seen in Fig 2.2.

When talking about polytopes, we may want to refer to their **dimension**. In order to understand dimension, we need to know about affine spaces.

**Definition 2.2.4.** An **affine space** \( W \) is a subset of \( \mathbb{R}^d \) that can be expressed as \( W = a + V \) for a vector subspace \( V \subseteq \mathbb{R}^d \) and some \( a \in \mathbb{R}^d \). The **dimension** of an affine space is the dimension of \( V \).
Figure 2.2: Top: Three halfspaces in \( \mathbb{R}^2 \). Bottom: The intersection of these halfspaces forms a polytope, the dotted triangle.

In simpler terms, an affine space is a translation of some vector space, and we want to know the smallest such space that can contain our polytope to know its dimension.

**Definition 2.2.5.** The **dimension** of a polytope \( P \) is the dimension of the smallest affine space \( W \) such that \( P \subseteq W \).

The generators we intend to use for our algebra are not in fact the polytopes themselves, but **indicator functions** on polytopes. Indicator functions only have two possible outputs, 1 or 0, based on some criteria. The indicator function on a polytope takes points in \( \mathbb{R}^d \) as
Figure 2.3: An indicator function works a bit like a metal detector.

input, and outputs 1 if the point is in the polytope, or 0 if it is not. More precisely, given a polytope $P \subseteq \mathbb{R}^d$, we define $f_P : \mathbb{R}^d \to \mathbb{R}$:

$$f_P(v) = \begin{cases} 
1 & \text{if } v \in P, \\
0 & \text{otherwise}. 
\end{cases}$$

**Definition 2.2.6.** A polytopal function on $\mathbb{R}^d$ is an $\mathbb{R}$-linear combination of indicator functions on polytopes in $\mathbb{R}^d$.

Let $\Pi_0(\mathbb{R}^d)$ denote the set of all polytopal functions on $\mathbb{R}^d$. We use the letter $\Pi$ which is Greek for “P,” as in “polytope.” Notice the subscript, which foreshadows some changes will be made to the algebra before we are fully ready to work with it.
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Notice that $\Pi_0(\mathbb{R}^d)$ is a vector space with the operations of scalar multiplication and addition inherited from its superset, $\{ f : \mathbb{R}^d \to \mathbb{R} \}$. This means that, for $f$ and $g \in \Pi_0(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we have

$$\lambda \cdot f : \mathbb{R}^n \to \mathbb{R} \quad (\lambda \cdot f)(v) = \lambda f(v)$$

and

$$f + g : \mathbb{R}^n \to \mathbb{R} \quad (f + g)(v) = f(v) + g(v).$$

We can multiply our functions by a constant or we can add them together. This is possible because the codomain of the indicator functions is $\mathbb{R}$, which is a vector space that permits both addition and scalar multiplication.

Even though the generators of $\Pi_0(\mathbb{R}^d)$ are indicator functions on polytopes, many of its elements are not. Instead, they are linear combinations of indicator functions on polytopes. For an example of an element of $\Pi_0(\mathbb{R}^d)$ that can not be expressed as the indicator function on a single polytope, refer to Figure 2.4.

A notable property of addition is that, for polytopes $P$ and $Q$, where $P \cup Q$ is also a polytope, we have

$$f_P + f_Q = f_{P \cup Q} + f_{P \cap Q}. \quad (2.1)$$
Figure 2.4: We have that $f_P$ (top left) and $f_Q$ (top right) are both polytopes and elements of $\Pi_0(\mathbb{R}^2)$. However, $f_P + f_Q$ (bottom) is also valid element of $\Pi_0(\mathbb{R}^2)$, even though it is not an indicator function on one polytope. Also note the labeled point in the image of $P \cup Q$. Since this point appears in both $P$ and $Q$, the value of $f_P + f_Q$ at this point will be 2, whereas the value of $f_P + f_Q$ is 1 everywhere else in $P \cup Q$.

We can see that this equation is true using a case-by-case analysis. Given $x \in P$ or $x \in Q$ but $x \notin P \cap Q$, we have $f_P(x) + f_Q(x) = 1 = f_{P \cup Q}(x) + f_{P \cap Q}(x)$. Similarly, for $y \in P \cap Q$, we have $f_P(y) + f_Q(y) = 2 = f_{P \cup Q}(y) + f_{P \cap Q}(y)$. For $z \notin P$, $z \notin Q$, we will have $f_P(z) + f_Q(z) = 0 = f_{P \cup Q}(y) + f_{P \cap Q}(z)$.

Note that equation (2.1) demonstrates that expressions of elements in $\Pi_0(\mathbb{R}^d)$ as linear combinations of indicator functions on polytopes are not unique. In other words, indicator functions do not form a basis of $\Pi_0(\mathbb{R}^d)$.
In order to make the vector space $\Pi_0(\mathbb{R}^d)$ an algebra, we still need to define a multiplication operation. For multiplication, we use something called Minkowski sum.

**Definition 2.2.7.** The Minkowski sum of two polytopes $P$ and $Q$ in $\mathbb{R}^d$ is

\[ P + Q = \{ a + b \mid a \in P, b \in Q \} . \]

Informally, we can think of the Minkowski sum of two polytopes as picking up one polytope by a vertex and dragging it along all other points of the other polytope, as seen in Fig 2.4.

We define multiplication on the indicator functions of two polytopes $P$ and $Q$

\[ f_P \times f_Q = f_{P+Q} \]

where $P + Q$ is the Minkowski sum of $P$ and $Q$. Be careful! Here we use $+$ between two polytopes to indicate their Minkowski sum. Elsewhere in the paper, we use $+$ between two functions to indicate the sum of two functions. Note that the multiplicative identity is the indicator function on the origin.

When $g \in \Pi_0(\mathbb{R}^d)$ is the linear combination of several indicator functions, multiplication extends linearly. If $g_1 = \sum_i \alpha_i f_{P_i}$ for some finite number of indicator functions $f_{P_i}$, and $g_2 = \sum_j \beta_j f_{Q_j}$ for some finite number of indicator functions $f_{Q_j}$, we have

\[ g_1 \times g_2 = \sum_{i,j} \alpha_i \beta_j \left( f_{P_i} \times f_{Q_j} \right) . \]

Because we know the expressions of elements in $\Pi_0(\mathbb{R}^d)$ are not unique, we must verify that multiplication is indeed well-defined. That is, when we multiply elements, we must
Figure 2.5: Constructing the Minkowski sum of a square and a triangle.
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Figure 2.6: These two polytopes may look the same to you, but in $\Pi_0(\mathbb{R}^2)$, they actually result in different indicator functions.

know that the outcome is always the same, regardless of how those elements are expressed as linear combinations of additive functions. Luckily this was proved in Lemma 6 of McMullen’s paper [2], as well as Proposition 3 in the paper of Pukhlikov and Khovanskii [5].

**Theorem 2.2.1.** [2], [5] $\Pi_0(\mathbb{R}^d)$ is an $\mathbb{R}$-algebra.

One property of $\Pi_0(\mathbb{R}^d)$ is that identical polytopes in different places in $\mathbb{R}^d$ are not considered equal. Refer to Fig 2.6 for such an example. This may pose some challenge, because we are more often concerned with the shape itself than its position, so we would want to consider these polytopes equivalent. To identify polytopes that differ only by translation, we introduce the translation ideal $T \subseteq \Pi_0(\mathbb{R}^d)$.

**Definition 2.2.8.** The translation ideal of $\Pi_0(\mathbb{R}^d)$ is defined to be
\( \mathcal{T} := \mathbb{R}\{f_{P+t} - f_P \mid P \subseteq \mathbb{R}^d \text{ is a polytope and } t \in \mathbb{R}^d \} \).

We can verify that \( \mathcal{T} \) is in fact an ideal of \( \Pi_0(\mathbb{R}^d) \). Multiplying any \( f_Q \) by an element of \( \mathcal{T} \) gives

\[
f_Q \times (f_{P+t} - f_P) = f_Q \times f_{P+t} - f_Q \times f_P = f_{(Q+P)+t} - f_{Q+P}
\]

and this product is also an element of the ideal.

When we take the quotient over this ideal, we obtain \([f_{P+t}] = [f_P]\). Since \( P + t \) is just the translation of the polytope \( P \) by some vector \( t \), the indicator function on any polytope is equal to the indicator function on its translation. Indeed, this will make our algebra much more manageable.

**Definition 2.2.9.** We define the **polytope algebra** \( \Pi(\mathbb{R}^d) \) to be the quotient

\[
\Pi(\mathbb{R}^d) = \Pi_0(\mathbb{R}^d)/\mathcal{T}.
\]

We use brackets to indicate elements of \( \Pi(\mathbb{R}^d) \) (e.g. \([f_P]\)).

Note that the multiplicative identity is now the indicator function on *any* single point.

### 2.3 Self-Destructive Tendencies

"Trouble" with a capital T and that rhymes with P and that stands for "polytope."

Luckily for us, the polytope algebra is extremely well-studied. Many properties are already known and proven. For example, consider the following specific property, which is Lemma 13 in "The Polytope Algebra" [2].
**CHAPTER 2. BACKGROUND**

**Theorem 2.3.1.** [2] For any polytope $P \subseteq \mathbb{R}^d$, we have

$$([f_P] - 1)^{d+1} = 0 \in \Pi(\mathbb{R}^d).$$

This makes $[f_P] - 1$ a **nilpotent element** of $\Pi(\mathbb{R}^d)$. A nilpotent element is some element that can be multiplied by itself some number of times until it is equal to 0.

---

**What’s Going on Here?**

Nilpotent elements are a fascinating thing to find in any mathematical structure, because they don’t appear in many familiar settings like the real numbers. That is, we will never multiply a nonzero number by itself and get 0 as a result, no matter how many times we repeat the multiplication. However, we see that if we multiply $[f_P] - 1$ by itself sufficiently many times, it simply disappears.

---

Proving Theorem 2.3.1 in the general-dimensional case can become quite knotty. However, since this paper will focus on an algebraic result in the one-dimensional case for normal complexes, we will focus on the proof of the one-dimensional case for polytopes.

**Theorem 2.3.2.** For a polytope $P \subseteq \mathbb{R}$,

$$([f_P] - 1)^2 = 0 \in \Pi(\mathbb{R}).$$
It’s hard not to empathize with $[f_p] - 1$.

Figure 2.7: Mathematical objects have feelings too.

Proof. A polytope in $\mathbb{R}$ consists of the closed line segment between two points. Call this line segment $[a, b]$. By translation, we have

$$[f_{[a,b]}] = [f_{[0,b-a]}].$$

Then consider

$$([f_{[0,b-a]}] - 1)^2 = [f_{[0,a-b]}]^2 - 2[f_{[0,b-a]}] + 1.$$
\[\left[f_{[0,b-a]}\right]^2 = \left[f_{[0,2(b-a)]}\right].\]

This is because the Minkowski sum of a line segment with itself is a line segment twice the previous length. Also note that by translation, \([f_{[0,b-a]}] = [f_{[b-a,2(b-a)]}].\) Therefore

\[2f_{[0,b-a]} = \left[f_{[0,b-a]}\right] + \left[f_{[b-a,2(b-a)]}\right].\]

Because \(f_P + f_Q = f_{P \cup Q} + f_{P \cap Q}\) (remember that nice property from Equation (2.1)?), we have

\[\left[f_{[0,b-a]}\right] + \left[f_{[b-a,2(b-a)]}\right] = \left[f_{[0,2(b-a)]}\right] + 1\]

Mind the overlap! Notice that the point at \(b-a\) is a “double point” because it is included in both line segments. In other words, the output of these functions at the point \(b-a\) is 2.

Then our result is

\[\left(\left[f_{[0,b-a]}\right] - 1\right)^2 = \left[f_{[0,b-a]}\right]^2 - 2\left[f_{[0,b-a]}\right] + 1\]
\[= \left[f_{[0,2(b-a)]}\right] - (\left[f_{[0,2(b-a)]}\right] + 1) + 1\]
\[= 0\]

as desired.

At first glance, this proof looks somewhat unsatisfying. Perhaps you hoped to develop a geometric understanding of \([f_P] - 1\). What you must remember is that \([f_P] - 1\) cannot
be expressed as the indicator function on any one polytope. It can only be expressed as the difference of indicator functions. We have not defined the Minkowski sum of two things that are not polytopes, so we can only find the product by taking the product as a linear combination of indicator functions on polytopes.
Chapter 3

Getting to Know Normal Complexes

Now that we have a basic understanding of the polytope algebra, we can begin to construct an analogue for normal complexes. Normal complexes are a recent introduction to the field of algebraic combinatorics from Nathanson and Ross in 2021 [3]. The simple mathematical description of a normal complex is that it is an orthogonally truncated fan. A fan is a bunch of unbounded polyhedra, stuck together at the origin. We apply a bound on the polyhedra, so the resulting object, the normal complex, is bounded. Many polytopes can be made using this “normal complex” construction, but we can also create many objects that are distinctly not polytopes, such as the two-dimensional normal complex seen in Figure 3.1.

3.1 A Recipe for Normal Complexes

*I could go on forever if I weren’t truncated.*
CHAPTER 3. GETTING TO KNOW NORMAL COMPLEXES

Figure 3.1: A two-dimensional normal complex.

To begin our more careful definition of a normal complex, we will return to polyhedral geometry. The first thing we will need is a cone.

Recall that a polytope is a bounded intersection of finitely many halfspaces. A polyhedron is similar, but it can be unbounded.

**Definition 3.1.1.** A polyhedron \( P \subseteq \mathbb{R}^d \) is an intersection of finitely many halfspaces.

We define the **dimension** of a polyhedron exactly as we did with the polytopes. It is the dimension of the smallest affine space containing it.

**Definition 3.1.2.** A cone \( \sigma \) is a polyhedron such that all defining halfspaces \( \mathcal{H}_{a_i, b_i} \) pass through the origin. Stated another way, we have that

\[
\sigma = \bigcap_{i=0}^{m} \mathcal{H}_{v_i, 0}.
\]

The cones we use to construct normal complexes must be **strongly convex**, also known as **pointed**. This means that \( \sigma \) does not contain a nontrivial vector space.
To combine multiple polyhedra together into a single structure, we must introduce the concept of a face.

**Definition 3.1.3.** A supporting hyperplane $H_{a,b}$ of a polyhedron $P$ is a hyperplane such that $P$ is contained completely in one of the halfspaces $H_{a,b}^+$ or $H_{a,b}^-$. 

**Definition 3.1.4.** A face of a polyhedron is the intersection of the polyhedron with some supporting hyperplane.

A one-dimensional face of a cone is called a ray. We call a cone simplicial if it has the same number of rays as its dimension. We denote the set of all rays in a cone $\sigma$ as $\sigma(1)$.

**Definition 3.1.5.** A polyhedral complex $C$ is a finite collection of polyhedra in $\mathbb{R}^d$ such that

1. For all polyhedra $P \in C$, each face of $P$ is also in $C$.

2. For any pair of polyhedra $P, Q \in C$, their intersection $P \cap Q$ is a face of both $P$ and $Q$.

**Definition 3.1.6.** A fan $\Sigma$ is a polyhedral complex consisting only of cones.

For the scope of this project, we would like our fans to be simplicial and pure. This will make our fans easier to work with.
• A fan $\Sigma$ is **simplicial** if all its cones are simplicial.

• A fan $\Sigma$ is **pure** if every maximal cone of $\Sigma$ has the same dimension.

We may want to refer to those cones of $\Sigma$ of only a specific dimension. We use $\Sigma(n)$ to denote the set of $n$-dimensional cones of $\Sigma$. Importantly, $\Sigma(1)$ refers to the rays of $\Sigma$. For each ray $\rho \in \Sigma(1)$, $u_\rho$ denotes the unit vector along $\rho$.

The last step in constructing a normal complex is truncating along each ray, so our resulting normal complex becomes bounded. In order to do this, we assign each ray $\rho$ a value $z_\rho \in \mathbb{R}_{\geq 0}$. The set of all such $z$-values associated with a normal complex is denoted $z \in \mathbb{R}^\Sigma (1)$. We use the hyperplane orthogonal to $\rho$ to generate a halfspace, $\mathcal{H}_{u_\rho, z_\rho}$, which will be used to truncate $\sigma$.

In other words, this $z$-value tells us how far away from the origin we want to truncate the ray. We need to choose these values carefully so nothing unexpected happens in the
structure of the normal complex. For our normal complex to be a polytopal complex, each truncated cone must meet its neighbors along faces. We use the following property to ensure this happens.

**Definition 3.1.7.** A choice of \( z \in \mathbb{R}^{\Sigma(1)} \) is **pseudocubical** with respect to a fixed fan \( \Sigma \) and a fixed inner product \( * \) if \( \bigcap_{\rho \in \sigma(1)} H_{u_\rho, z_\rho} \subseteq \sigma \), for all \( \sigma \in \Sigma \). The set of pseudocubical choices of \( z \) is denoted \( \text{Cub}(\Sigma, *) \subseteq \mathbb{R}^{\Sigma(1)} \). We can see an example of choices of cubical, pseudocubical, and non-pseudocubical \( z \)-values for a two-dimensional cone in Figure 3.4.

**Example 3.1.1.** In Figure 3.5, we can see a one-dimensional fan \( \Sigma \) defined by \( u_0 = (1, 0) \), \( u_1 = (0, 1) \), and \( u_2 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \). In the one-dimensional case, the cones are all rays, but they can exist in any dimensional space. Then, we see a choice of pseudocubical \( z \)-values for \( \Sigma \). Notice that for a one-dimensional \( \Sigma \), \( z \in \text{Cub}(\Sigma, *) \) only requires that \( z_\rho \geq 0 \) for all \( \rho \in \Sigma \). This is because each maximal cone only has a single ray, so any positive \( z \)-value will truncate the cone within the cone.
Figure 3.5: A one-dimensional fan living in $\mathbb{R}^2$ (left), and an orthogonal truncation of that fan, using pseudocubical $z$-values (left).

**Definition 3.1.8.** Let $\Sigma$ be a fan in $\mathbb{R}^d$, $\ast$ an inner product on $\mathbb{R}^d$, and $z \in \overline{\text{Cub}}(\Sigma, \ast)$. The normal complex $C_{\Sigma, \ast}(z)$ is the collection of the polytopes

$$\sigma \cap \bigcap_{\rho \in \sigma(1)} \mathcal{H}_{\tilde{z}_\rho},$$

for all $\sigma \in \Sigma$.

**Theorem 3.1.1.** [3] Given a pseudocubical $z$-vector, $z \in \overline{\text{Cub}}(\Sigma, \ast)$, $C_{\Sigma, \ast}(z)$ is a polytopal complex.

Fig. 3.1 shows a two-dimensional normal complex living in $\mathbb{R}^3$. Notice how the cones are truncated so that each of polytopes meets the other at a face. We contrast the normal complex with its fan in Fig 3.6.
With normal complexes defined, we can now organize them into an algebra, following the example of the polytope algebra.

### 3.2 Building A New Algebra

"I want to make something good. / I want to make something better. / Something that cannot leave the ground, / Unless we lift it up together." -Modern Baseball, “Note to Self”

The first step in creating our new algebra of normal complexes is to fix a fan $\Sigma$, and limit our algebra to normal complexes built from $\Sigma$. We also require a fixed choice of inner product, $\ast$, in order to truncate the fan. To keep this clear, we will call the initial form of the algebra of normal complexes $\Pi_0(\Sigma)$.

Recall that in the previous chapter, our generators were not the polytopes themselves, but instead the indicator functions of polytopes. Similarly, here, the generators of $\Pi_0(\Sigma)$ will
not be the normal complexes themselves, but indicator functions on them. Let \( N = C_{\Sigma, *}(z) \) be a normal complex.

\[
f_N(v) = \begin{cases} 
  1 & \text{if } v \in N, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Definition 3.2.1.** We define \( \Pi_0(\Sigma) \) to be \( \mathbb{R}\{f_N \mid N = C_{\Sigma, *}(z) \text{ for some } z \in \text{Cub}(\Sigma, *)\} \)

Since indicator functions on normal complexes are still just functions from \( \mathbb{R}^d \to \mathbb{R} \), we can still do addition and scalar multiplication the same way we did in the polytope algebra.

However, we must define multiplication differently than we did in the polytope algebra, because . For normal complexes \( N \) and \( M \) with associated \( z \)-vectors \( z_N \) and \( z_M \), we define \( f_N \times f_M \) to be the indicator function on the normal complex with associated \( z \)-vector \( z_N + z_M \). As with polytopes, we allow multiplication to extend linearly.

**Definition 3.2.2.** Let \( g_1 \) and \( g_2 \) be elements of \( \Pi_0(\Sigma) \). If \( g_1 = \sum_i \alpha_i f_{C_{\Sigma, *}(z_i)} \) and \( g_2 = \sum_j \beta_j f_{C_{\Sigma, *}(z_j)} \), then their **product** is defined

\[
g_1 \times g_2 = \sum_{i,j} \alpha_i \beta_j f_{C_{\Sigma, *}(z_i + z_j)}
\]

We ensure that \( g_1 \times g_2 \) is a valid element of \( \Pi_0(Sigma) \) by verifying each of its component normal complexes still have pseudocubical \( z \)-values, using a proposition of Nathanson and Ross.

**Proposition 3.2.1.** [3] For two \( z \)-vectors \( z_N, z_M \in \text{Cub}(\Sigma, *) \), it will be the case that \( z_N + z_M \in \text{Cub}(\Sigma, *) \).
CHAPTER 3. GETTING TO KNOW NORMAL COMPLEXES

Recall the a normal complex is also a polytopal complex, and its components are just polytopes. Nowak, O’Melveny, and Ross showed that we can interpret the product \( f_N \times f_M \) as the indicator function on the cone-wise Minkowski sum of each polytope in the complex \[4\]. If \( N \) and \( M \) are normal complexes with indicator functions \( f_N \) and \( f_M \) then \( f_{N+M} \) refers to the indicator function on \( C_{\Sigma,*}(z_N + z_M) \), which is also the Minkowski sum of the polytopes of the complex.

**Theorem 3.2.1.** The multiplication operator on \( \Pi_0(\Sigma) \) is well-defined, and \( \Pi_0(\Sigma) \) forms an \( \mathbb{R} \)-algebra.

**Proof.** Let \( f, g \in \Pi_0(\Sigma) \), and let \( f' \) and \( g' \) be alternate representations of \( f \) and \( g \). That is, \( f' = f \) and \( g' = g \), as functions \( \mathbb{R}^d \to \mathbb{R} \). Consider \( f \) and \( g \) restricted to a single cone \( \sigma \). Then \( f|_{\sigma} \) and \( g|_{\sigma} \in \Pi_0(\mathbb{R}^d) \). It’s also true that \( f|_{\sigma} = f'|_{\sigma} \) and \( g|_{\sigma} = g'|_{\sigma} \). We know that multiplication in \( \Pi_0(\mathbb{R}^d) \) is well-defined from Theorem 2.2.1. Therefore, \( f|_{\sigma} \times g|_{\sigma} = f'|_{\sigma} \times g'|_{\sigma} \) for all cones \( \sigma \in \Sigma \). Since multiplication of normal complexes is equivalent to the cone-wise Minkowski sum, we conclude that \( f \times g = f' \times g' \). Thus, multiplication is well-defined in \( \Pi_0(\Sigma) \).

We now verify that \( \Pi_0(\Sigma) \) satisfies the axioms of an \( \mathbb{R} \)-algebra. Recall that these axioms are as follows:

1. \( (x + y) \times z = x \times z + y \times z \)
2. \( z \times (x + y) = z \times x + z \times y \)
3. \((ax) \times (by) = (ab)(x \times y)\)

Let \(g_1 = \sum_i \alpha_i f_{C_\Sigma,*}(z_i)\), \(g_2 = \sum_j \beta_j f_{C_\Sigma,*}(z_j)\), and \(g_3 = \sum_k \gamma_k f_{C_\Sigma,*}(z_k)\). Then

\[
(g_1 + g_2) \times g_3 = \left( \sum_i \alpha_i f_{C_\Sigma,*}(z_i) + \sum_j \beta_j f_{C_\Sigma,*}(z_j) \right) \times \sum_k \gamma_k f_{C_\Sigma,*}(z_k)
\]

\[
= \sum_{i,k} \alpha_i \gamma_k f_{C_{\sigma,*}}(z_i + z_k) + \sum_{j,k} \beta_j \gamma_k f_{C_{\sigma,*}}(z_j + z_k)
\]

\[
= \left( \sum_i \alpha_i f_{C_\Sigma,*}(z_i) \right) \times \left( \sum_k \gamma_k f_{C_\Sigma,*}(z_k) \right) + \left( \sum_j \beta_j f_{C_\Sigma,*}(z_j) \right) \times \left( \sum_k \gamma_k f_{C_\Sigma,*}(z_k) \right)
\]

\[
= g_1 \times g_3 + g_2 \times g_3.
\]

We can see that (1) holds. Therefore, (2) also holds, because + and \(\times\) are commutative.

Finally, we know that (3) holds, because

\[
ag_1 \times bg_2 = a \sum_i \alpha_i f_{C_\Sigma,*}(z_i) \times b \sum_j \beta_j f_{C_\Sigma,*}(z_j)
\]

\[
= ab \sum_{i,j} \alpha_i \beta_j f_{C_\Sigma,*}(z_i + z_j)
\]

\[
= ab(g_1 \times g_2).
\]

We conclude that \(\Pi_0(\Sigma)\) is an \(\mathbb{R}\)-algebra. \(\square\)

Here, our multiplicative identity remains the indicator function on the origin, which can be represented as the normal complex where the \(z\)-values are all zeroes.

Next, we construct the translation ideal. Unlike the translation ideal on polytopes, we cannot translate the entire normal complex. By definition, the cones \textit{must} go through the origin. Instead, we will translate the truncating hyperplanes as a unit, like moving a
Figure 3.7: We can imagine translating a normal complex like moving a magnifying glass over a page.

magnifying glass over text on a page. The text, which represents the cones, does not change, nor does the shape and size of the lens itself, but what part we can see in our lens, our normal complex, changes.

Suppose we want to translate a hyperplane $H_{u_\rho, z_\rho}$ by some vector $t \in \mathbb{R}^d$. We will define a new hyperplane by simplifying the equation $(v - t) * u_\rho = z_\rho$ to obtain $v * u_\rho = z_t^\rho$, where $z_t^\rho = z_\rho + t * u_\rho$. Then the hyperplane $H_{u_\rho, z_t^\rho}$ represents the hyperplane translated by $t$.

A translate of a normal complex $N = C_{\Sigma, *}(z)$ by a vector $t \in \mathbb{R}^d$ is

$$N + t := C_{\Sigma, *}(z^t),$$

as long as $z^t \in \overline{\text{Cub}}(\Sigma, *)$.

Notice that this requires that the new $z^t$-values are also pseudocubical. This will ensure $N + t$ is itself a normal complex.
**Definition 3.2.3.** The translation ideal of $\Pi_0(\Sigma)$ is defined to be

$$\mathcal{T} := \mathbb{R}\{f_{N+t} - f_N \mid N = C_{\Sigma,*}(z), t \in \mathbb{R}^d \text{ such that } z, z^t \in \overline{\text{Cub}}(\Sigma,*)\}.$$ 

**Proposition 3.2.2.** $\mathcal{T}$ is an ideal of $\Pi_0(\Sigma)$. 

**Proof.** Suppose we have two normal complexes $N = C_{\Sigma,*}(z_N)$ and $M = C_{\Sigma,*}(z_M)$. Consider when we multiply an element of the translation ideal by a general element of the algebra

$$(f_{N+t} - f_N) \times f_M = f_{N+t} \times f_M - f_N \times f_M.$$ 

We have that $f_N \times f_M$ is the indicator function on $C_{\Sigma,*}(z_N + z_M)$, which is $f_{N+M}$. Similarly, $f_{N+t} \times f_M$ is the indicator function on $C_{\Sigma,*}(z_N^t + z_M)$, which is $f_{N+t+M}$. The result is

$$f_{N+t} \times f_M - f_N \times f_M = f_{N+M+t} - f_{N+M}$$

which is an element of the ideal. $\square$

As with the polytope algebra, the final form of the algebra of normal complexes is a quotient.

**Definition 3.2.4.** The algebra of normal complexes is defined

$$\Pi(\Sigma) = \Pi_0(\Sigma)/\mathcal{T}.$$ 

We denote the generators of $\Pi(\Sigma)$ using brackets (e.g. $[C_{\Sigma,*}(z)]$). For simplicity, we will not use $f$, but our elements remain cosets of indicator functions.

Our hope as we continue investigating $\Pi(\Sigma)$ is that it resembles $\Pi(\mathbb{R}^d)$ in some important ways.
Chapter 4

A Nilpotency in the Algebra for Normal Complexes

All this setup has been leading to this paper’s original results, which is that $([N] - 1)^2 = 0$ holds in $\Pi(\Sigma)$ for a one-dimensional fan $\Sigma$. It is an analogue to the nilpotency we discussed in the polytope algebra. While the generalized version of the proof in the second section of this chapter demonstrates the actual result, the first section contains an example that readers may find illuminating when trying to understand the ideas in the second section. Specifically, the pictures in the first section offer a guide for how to interpret all of the translations and differences of the normal complexes.
4.1 An Illustrative Example

*An example is worth a thousand definitions.*

Now that we have set up our algebra, we can begin by demonstrating that for a one-dimensional normal complex $N$, $([N] - 1)^2 = 0 \in \Pi(\Sigma)$. Here is our strategy: We start by expanding $([N] - 1)^2 = [N]^2 - 2[N] + 1$. The first term of the expanded equation $[N]^2$ represents a version of $N$ where each segment is twice as long. We want to manipulate the middle term, $2[N]$, so that it is also equivalent to a version of $N$ where each segment is twice as long (plus hopefully an extra point, to cancel out $+1$). To do this, we separate $N$ into its cones and the origin. We move each cone outward by the length of that line segment. Then we combine this “blown-up” normal complex with an unaltered $N$, to create a double-length version of $N$, as desired. The origin from the first normal complex cancels out the $+1$ term, so that the expression equals 0.

Now, we will execute the plan in more detail. Let us begin with a simple example of a one-dimensional fan $\Sigma \subseteq \mathbb{R}^2$ with cones $\rho_0$, $\rho_1$, and $\rho_2$ generated by the unit vectors $u_0 = (1, 0)$, $u_1 = (0, 1)$, and $u_2 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. Because $\Sigma$ is pure and one-dimensional, each maximal cone consists of a single ray.

There are three $z$-values, $z_0$, $z_1$, and $z_2$. We will use the values $z_0 = 2$, $z_1 = \frac{3}{2}$, and $z_2 = 1$. Then we have $N = C_{\Sigma,*}(2, \frac{3}{2}, 1)$. Now consider $([N] - 1)^2$. We obtain


Notice that $[N]^2$ is the product of a normal complex with itself, which means we will...
double all the $z$-values. Thus, each line segment of $[N]^2$ is twice as long as the line segments of $[N]$.

As we begin working with equations, we will use slightly simplified notation to make it easier to follow what is happening. When we write $N$ or $C_{\Sigma,*}(z)$, we are actually referring to coset of the indicator functions on these normal complexes in $\Pi(\Sigma)$.

If $N = C_{\Sigma,*}(2, \frac{3}{2}, 1)$, then $N^2 = C_{\Sigma,*}(4, 3, 2)$.

Then, observe that it is possible to split a single normal complex into the sums and differences of several normal complexes. For example,

$$C_{\Sigma,*}\left(2, \frac{3}{2}, 1\right) - C_{\Sigma,*}\left(0, \frac{3}{2}, 1\right)$$

will result in the indicator function on the rightmost segment of the normal complex, without the origin. We can see what this indicator function represents in Figure 4.2.
In this way, we can construct each of the cones of $N$, minus the origin. Another way to think of this is as $N$ restricted to the relative interior of a ray $\rho^o$. Adding the origin back in, we obtain $N$. For a one-dimensional normal complex, pseudocubical $z$-values must be greater than or equal to 0. We must also have that the $z$-values after translation, $z^t$ are still pseudocubical. We set any “cancelled” $z$-value to $\max(z_\rho)$. This is sufficient to guarantee that when we translate each line segment by $z_\rho u_\rho$, the $z$-vector will still be pseudocubical. In this case, $\max(z_\rho) = 2$. We rewrite $N$ as follows:

\[
N = C_{\Sigma,*}(0, 0, 0) + C_{\Sigma,*}(2, 2, 2) - C_{\Sigma,*}(0, 2, 2) + C_{\Sigma,*}(2, 2, 2) - C_{\Sigma,*}(0, 2, 2) + C_{\Sigma,*}(2, 2, 1) - C_{\Sigma,*}(2, 2, 0) + C_{\Sigma,*}(2, 2, 2) - C_{\Sigma,*}(0, 2, 2) - C_{\Sigma,*}(2, 2, 0).
\]

Next, we translate $C_{\Sigma,*}(2, 2, 2) - C_{\Sigma,*}(0, 2, 2)$ in the direction of $u_0$ by a distance $z_0$. We obtain

\[
C_{\Sigma,*}(2, 2, 2) - C_{\Sigma,*}(0, 2, 2) = C_{\Sigma,*} \left(4, 2, 2 - \sqrt{2}\right) - C_{\Sigma,*} \left(2, 2, 2 - \sqrt{2}\right).
\]

We repeat this process, translating $N|_{\sigma_1}$ in the direction of $u_1$ by $z_1$, and $N|_{\sigma_2}$ in the direction of $u_2$ by $z_2$ to obtain

\[
C_{\Sigma,*} \left(2, \frac{3}{2}, 2\right) - C_{\Sigma,*} \left(2, 0, 2\right) = C_{\Sigma,*} \left(2, 3, \frac{3\sqrt{2}}{4}\right) - C_{\Sigma,*} \left(2, \frac{3}{2}, \frac{3\sqrt{2}}{4}\right)
\]

\[
C_{\Sigma,*} \left(2, 2, 1\right) - C_{\Sigma,*} \left(2, 2, 0\right) = C_{\Sigma,*} \left(2 - \frac{\sqrt{2}}{2}, 2 - \frac{\sqrt{2}}{2}, 2\right) - C_{\Sigma,*} \left(2 - \frac{\sqrt{2}}{2}, 2 - \frac{\sqrt{2}}{2}, 1\right).
\]
We can see all these pieces in one picture in Fig 4.2. Recall that this is just a fancy reinterpretation of \([N] \in \Pi(\Sigma)\). If we add the indicator function on these terms with \(N\), we see that we actually obtain \([N]^2 + 1\). To visualize this, imagine pasting the normal complex in Fig 4.1 on top of the blown up normal complex on the right in Fig 4.2. The result would be a normal complex where each line segment is twice as long.

Referring back to our original equation, we have \([N]^2 - 2([N]) + 1 = [N]^2 - ([N]^2+1) + 1 = 0\), as desired.

This example offers a good idea of how we might make a general proof for any one-dimensional normal complex \(\Sigma\). In the following section, we make that general proof explicit.
Figure 4.3: Our feelings have the potential to influence how we see mathematics.
What’s Going on Here?

The best way to understand this proof is to look at the pictures. Start with Fig. 4.1 to see what our normal complex looks like. The expression $N^2 - 2N + 1$ consists of one “double-length” normal complex, two negative normal complexes, and one point. The bulk of the proof involves carefully manipulating the middle term, $-2N$, so it becomes $N^2 + 1$. We break $N$ into its component line segments. One of these line segments can be seen on the left in Fig 4.2. We translate each line segment outward along its ray, by the same distance as each line segment is long. The result is seen on the right of Fig 4.2. Then it is possible to fit $N$ perfectly in the middle, like a puzzle piece. This creates a double-length $N$ along with an extra point, as desired. We obtain $N^2 - 2N + 1 = 0$. In the next section, we follow many of the same steps to show that this process works for any one-dimensional complex.

4.2 The One-Dimensional Proof

A one-dimensional story as told by a multi-dimensional mathematician.

We can draw from the example in the previous section to understand how the proof might work in the general case of a one-dimensional normal complex.
Theorem 4.2.1. For a one-dimensional fan $\Sigma$, whenever $[N] \in \Pi(\Sigma)$,

$$([N] - 1)^2 = 0.$$ 

Proof. Let $N = C_{\Sigma,*}(z)$, where $\Sigma$ is a one-dimensional fan and $z \in \overline{\text{Cub}(\Sigma,*)} = \mathbb{R}_{\geq 0}^{\Sigma(1)}$. Let $z_{\text{max}} = \max\{z_\rho \mid \rho \in \Sigma(1)\}$.

Notice that $([N] - 1)^2 = [N]^2 - 2[N] + 1$. Consider one of the $[N]$ terms in the middle. We can express $[N]$ as the sum of its segments, plus the origin. In order to isolate a segment, define

$$N|_{\rho^o} := C_{\Sigma,*}(z_{\text{max}}, z_{\text{max}}, \ldots, z_\rho, z_{\text{max}}, \ldots) - C_{\Sigma,*}(z_{\text{max}}, z_{\text{max}}, \ldots, 0, z_{\text{max}}, \ldots),$$ 

where $z_\rho$ is the $z$-value associated with the single ray $\rho$. In other words, we express the isolated line segment $\sigma$ as the difference of two normal complexes. The $\rho^o$ refers to the interior of the ray, so $N|_{\rho^o}$ is the restriction of the normal complex $N$ to the interior of the ray $\rho$. All $z$-values are $z_{\text{max}}$ in both normal complexes, except the $\rho^{\text{th}}$ $z$-value, which is $z_\rho$ in the first normal complex, and $0$ in the second.

Once we have expressed $N|_{\rho^o}$ this way, we scooch both normal complexes over by a distance $z_\rho$ in the direction of $u_\rho$. Since $u_\rho$ is a unit vector in the direction of $\rho$, we need only translate by $z_\rho(u_\rho)$. When translated, the $z_{\text{max}}$ associated with any other ray $\mu \in \sigma(1)$ in both terms will become $z_{\text{max}} + z_\rho(u_\rho \ast u_\mu)$. As we translate, we want to make sure the $z$-values remain pseudocubical. Luckily in the one-dimensional case, the pseudocubical condition only requires that each $z$-value be greater than or equal to $0$. Notice that $z_{\text{max}} + z_\rho(u_\rho \ast u_\mu) \geq 0$, \ldots
because $|u_{\rho} \ast u_{\mu}| \leq 1$ and $z_{\text{max}} \geq z_{\rho}$ for all $z_{\rho}$. The new difference of indicator functions is

$$N|_{\rho^\circ} = C_{\Sigma,*} (z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_0), z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_1), ... z_{\rho} + z_{\rho}(u_{\rho} \ast u_{\rho}), ... ) -$$

$$C_{\Sigma,*} (z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_0), z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_1), ... z_{\rho}(u_{\rho} \ast u_{\rho}), ... )$$

$$= C_{\Sigma,*} (z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_0), z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_1), ... 2z_{\rho}, ... ) -$$

$$C_{\Sigma,*} (z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_0), z_{\text{max}} + z_{\rho}(u_{\rho} \ast u_1), ... z_{\rho}, ... ).$$

Because the rest of the $z$-vector is the same, the value of this difference of indicator functions will be nonzero only along $\rho$ at a distance $z_{\rho} < d \leq 2z_{\rho}$ from the origin, where the value is 1.

After isolating and translating the individual cones, do not forget that we still have a $+1$ term that represents the origin of this normal complex.

We can describe $[N]$ as an indicator function that is equal to 1 at a distance less than or equal to $z_{\rho}$ along each $\rho$, and 0 elsewhere. Adding the isolated and translated version of the normal complex, expressed as the difference above, with $[N]$, we have an indicator function that is equal to 1 at a distance less than or equal to $2z_{\rho}$ along each $\rho$. This is equivalent to $[C_{\Sigma,*}(2\rho)] = [N^2]$. Thus, we have a positive $[N]^2$ in the first term, and we can manipulate
the middle term.

\[
2[N] = [N] + 1 + \sum_{\rho \in \Sigma(1)} N|_{\rho}
\]

\[
= [N] + 1 + \sum_{\rho \in \Sigma(1)} N_{\rho} + z_{\rho} u_{\rho}
\]

\[
= [N]^2 + 1
\]

where \( N_{\rho} + z_{\rho} u_{\rho} \) is the difference of normal complexes representing the segment along \( \rho \) translated by \( z_{\rho} u_{\rho} \).

The origin in the last term cancels with the origin in the negative middle term, and we have 0, as desired. \qed
Chapter 5

Where Will We Go Next?

Early in this paper, we discussed the usefulness of putting mathematical objects into algebraic structures, like algebras, in order to better understand their properties. So far, we have explored the properties of the algebra of normal complexes, and we have found ways to empathize with mathematical objects and engage in poetic ideas about what these results might mean. The question is then, “What are the next steps?”

The first and most obvious extension of Theorem 4.2.1 is to prove the general-dimensional case, which has already been proven for the polytope algebra. The significance of knowing that \((P - 1)^{d+1} = 0\) holds for all \(P \in \Pi(\mathbb{R}^d)\) is that it allows us to apply a grading to \(\Pi(\mathbb{R}^d)\).

**Definition 5.0.1.** A **graded algebra** is an algebra \(A\) that can be decomposed into the direct sum of vector spaces \(V_i\) such that \(V_i V_j \subseteq V_{i+j}\). This decomposition is referred to as a grading.
CHAPTER 5. WHERE WILL WE GO NEXT?

With the property that \((P - 1)^{d+1} = 0\), we can actually define the logarithm of \([P]\) as

\[
\log([P]) = \log(1 + ([P] - 1)) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} ([P] - 1)^k.
\]

This will necessarily terminate because we know that \((P - 1)^{d+1} = 0\).

Finally, McMullen tells us the following about the polytope algebra. Define

\[
\Pi_k := \mathbb{R} \{ \log([P])^k \}
\]

Then we have the grading.

**Theorem 5.0.1.** [2] The polytope algebra is a graded algebra, with \(\Pi(\mathbb{R}^d) = \Pi_0 \oplus \Pi_1 \oplus \ldots \oplus \Pi_d\).

This theorem from the land of polytopes provides some inspiration for what we might be able to prove about the algebra of normal complexes. If we can prove the nilpotency holds in the algebras of higher dimensional normal complexes, then we may be able to prove that the algebra of normal complexes is also graded. The author is continuing to work with Dusty Ross and Ian Wallace to explore the general-dimensional cases of normal complexes.
What’s Going on Here?

When reading mathematics, it is generally valuable to orient oneself in the broader field. Why is this contribution interesting? What work can be done next? However, the lessons of doing mathematics extend beyond the results of the mathematics itself. Some of the takeaways and ongoing questions that arose from the construction of this thesis, and may be interesting for the reader to consider, are:

• How can we share our vulnerabilities about doing math?

• How can we have empathy when we do math?

• How can we share ourselves when we communicate mathematics?
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